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CFT, BCFT, ADE and all that

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These pedagogical lectures present some material, classical or more recent, on (Rational) Conformal Field Theories and their general setting "in the bulk" or in the presence of a boundary. Two well posed problems are the classification of modular invariant partition functions and the determination of boundary conditions consistent with conformal invariance. It is shown why the two problems are intimately connected and how graphs –ADE Dynkin diagrams and their generalizations– appear in a natural way.

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0. Introduction

These lectures aim at presenting some curious features encountered in the study of 2 D conformal field theories. The key words are graphs, or Dynkin diagrams, and indeed we shall encounter new avatars of the ADE Dynkin diagrams, and some generalizations thereof. The first lecture is devoted to a lightning review of Conformal Field Theory (CFT), essentially to recall essential notions and to establish basic notations. The study of modular invariant partition functions for theories related to the simplest Lie algebra $sl(2)$ leads to an ADE classification, as has been known for more than ten years. A certain frustration comes from the fact that we have no good reason to explain why this ADE classification appears, or no definite way to connect it to another existing classification (Lecture 2). Or, which amounts to the same, we have too many: depending on the way we look at these $sl(2)$ theories –their topological counterparts, their lattice realization–the reason looks different. Moreover, when we turn to higher rank algebras, the situation is even more elusive: it had been guessed long ago that classification should involve again graphs, though for reasons not very well understood, and a list of graphs had been proposed in the case of $sl(3)$. Recent progress has confirmed these expectations: as will be discussed in the final lecture 3, through the study of boundary conditions we now understand why graphs are naturally associated with CFTs and which properties they must satisfy. In the case of $sl(2)$, this leads in a straightforward way (but up to a little subtlety) to the ADE diagrams. Moreover, the classification of the graphs relevant in the case of $sl(3)$ has just been completed, see A. Ocneanu's lectures at this school.

1. A crash course on CFT

This section is devoted to a fast summary of concepts and notations in conformal field theories (CFTs).

1.1. On CFTs

A conformal field theory is a quantum field theory endowed with covariance properties under conformal transformations. We shall restrict our attention to two dimensions, and in a first step to the Euclidean plane, where the conformal transformations are realized by any analytic change of coordinates $z \mapsto \zeta(z)$, $\bar z \mapsto \bar\zeta(\bar z)$. In fact, it appears that the contributions of the variables z and $\bar z$ decouple [1] and that z and $\bar z$ may be regarded as independent variables. The zz component of the energy-momentum tensor $T_{\mu\nu}$ is analytic as a consequence of its tracelessness and conservation, and denoted $T(z)$, $T(z) := T_{zz}(z)$, and likewise $\bar T(\bar z) := T_{\bar z\bar z}(\bar z)$ is antianalytic. $T(z)$ is the generator of the infinitesimal change $z \mapsto \zeta = z + \epsilon(z)$ and likewise $\bar T(\bar z)$ for $\bar z \mapsto \bar\zeta(\bar z)$, in the sense that under such a change a correlation function of fields undergoes the change

$$\delta\langle\phi_{i_1}(z_1,\bar z_1)\cdots\phi_{i_n}(z_n,\bar z_n)\rangle = \frac{1}{2\pi i}\oint_{\mathcal C} \mathrm{d}w\,\epsilon(w)\,\langle T(w)\phi_{i_1}(z_1,\bar z_1)\cdots\phi_{i_n}(z_n,\bar z_n)\rangle + \mathrm{c.c.} \quad (1.1)$$

with an integration contour encircling all points $z_1,\cdots,z_n$.

Fields of a CFT are assigned an explicit transformation under these changes of variables. In particular, *primary fields* transform as $(h,\bar h)$ forms, i.e. according to

$$\tilde\phi(\zeta,\bar\zeta)=\Big(\frac{\partial z}{\partial\zeta}\Big)^h\Big(\frac{\partial\bar z}{\partial\bar\zeta}\Big)^{\bar h}\phi(z,\bar z) \tag{1.2}$$

where the real numbers $(h,\bar h)$ are the *conformal weights* of the field ϕ (see below for a representation-theoretic interpretation). For an infinitesimal change, $\zeta=z+\epsilon(z)$, $\bar\zeta=\bar z+\bar\epsilon(\bar z)$, if we set $\tilde\phi(z,\bar z)=\phi(z,\bar z)-\delta\phi(z,\bar z)$

$$\delta\phi=(h\epsilon'+\epsilon\partial_z)\phi+\text{c.c.} \tag{1.3}$$

Here and in the following, "c.c." denotes the formal complex conjugate, $(z,h)\to(\bar z,\bar h)$, "formal" because $\bar z$ is at this stage independent of z and h and $\bar h$ are also a priori independent. When the condition that $\bar z=z^*$ is imposed, $h+\bar h$ turns out to be the scaling dimension of the field ϕ and $h-\bar h$ its spin: locality (singlevaluedness of the correlators) imposes only on $h-\bar h$ to be an integer or half-integer.

In contrast to primary fields, the change of $T(z)$ itself has the form

$$\tilde T(\zeta)=\Big(\frac{\partial z}{\partial\zeta}\Big)^2T(z)+\frac{c}{12}\{z,\zeta\} \tag{1.4}$$

where $\{z,\zeta\}$ denotes the schwarzian derivative

$$\{z,\zeta\}=\frac{\frac{\partial^3z}{\partial\zeta^3}}{\frac{\partial z}{\partial\zeta}}-\frac{3}{2}\left(\frac{\frac{\partial^2z}{\partial\zeta^2}}{\frac{\partial z}{\partial\zeta}}\right)^2 \tag{1.5}$$

and where the parameter c in front of the anomalous schwarzian term is the *central charge* (of the Virasoro algebra, to come soon!). In other words, T transforms almost as a primary field of conformal weights $(h,\bar h)=(2,0)$ –a conserved current of scaling dimension 2– up to the schwarzian anomaly.

Exercise : derive the form of the variation of T under an infinitesimal transformation, i.e. the analog of (1.3).

In the spirit of local field theory, we assume that correlation functions as above are well defined in the complex plane with possible singularities only at coinciding points $z_i=z_j$ or $w=z_i$. Close to these points, there is a short distance expansion of products of fields. As an exercise, using Cauchy theorem, show that equation (1.3) (and its analog for T) may be rephrased as a statement on the expansions

$$T(w)\phi(z,\bar z)=\frac{h\phi(z,\bar z)}{(w-z)^2}+\frac{\partial\phi(z,\bar z)}{(w-z)}+\text{regular} \tag{1.6a}$$

$$T(w)T(z)=\frac{\frac{c}{2}}{(w-z)^4}+\frac{2T(z)}{(w-z)^2}+\frac{\partial T(z)}{(w-z)}+\text{regular} \tag{1.6b}$$

with similar expressions for the products with $\bar{T}(\bar{z})$. Eqs (1.6) are meant to describe the singular behaviour of correlation functions $\langle T(w)\phi(z,\bar{z})\cdots\rangle$, $\langle T(w)T(z)\cdots\rangle$, in the presence of spectator fields, as $z \to w$.

As usual in Quantum Field Theory, it is good to have two dual pictures at hand: the one dealing with correlation (Green) functions, as we have done so far, and in the spirit of quantum mechanics, the operator formalism, which describes the system by "states", i.e. vectors in a Hilbert space. In CFT, it is appropriate to think of a radial quantization in the plane: surfaces of equal "time" are circles centered at the origin, and the Hamiltonian is the dilatation operator. The origin in the plane plays the rôle of remote past, the remote future lies on the circle at infinity. On any circle, there is a description of the system in terms of a Hilbert space $\mathcal{H}$ of states, on which field *operators* act. If we expand the energy momentum tensor on its Laurent modes

$$T(z) = \sum_{n=-\infty}^{\infty} z^{-n-2} L_n \tag{1.7}$$

it is a good exercise (making use again of Cauchy theorem) to check that the expansion (1.6b), now regarded as an "operator product expansion" (OPE), may be rephrased as a commutation relation between the L's

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0} \ . \tag{1.8}$$

This is the celebrated **Virasoro algebra**, in which the *central charge* c appears indeed as the coefficient of the central term. Note that L_0 is the generator of dilatations, L_{-1} the one of translations. Together with L_1, they form a subalgebra.

What is the interpretation of L_1 and of that subalgebra?

Exercise : derive the commutation relation of L_n with the field operator ϕ, as a consequence of the equation (1.6a).

1.2. Extension to other chiral algebras.

A frequently encountered situation in CFT is that there is a larger ("extended") *chiral* algebra $\mathcal{A}$ encompassing the Virasoro algebra, and acting on fields of the theory. The latter thus fall again in representations of $\mathcal{A}$. The most common cases are those involving a current algebra, i.e. the affine extension $\hat{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$, or the so-called W algebras, or the various superconformal algebras, etc.

In the affine algebra $\hat{\mathfrak{g}}$, the important objects from our standpoint are the current $J(z)$ or its moments J_n, with values in the adjoint representation of $\mathfrak{g}$. In some basis, they satisfy the commutation relation

$$[J_n^a, J_m^b] = i f^{ab}{}_c J^c_{n+m} + \hat{k}\, n\, \delta_{n+m,\,0}\, \delta_{ab} \ , \tag{1.9}$$

with $\hat{k}$ a central element.

The Virasoro algebra is either a proper subalgebra (for ex, in the superconformal cases), or contained in the enveloping algebra of this chiral algebra. For example, in affine algebras,

the energy momentum tensor and Virasoro generators are obtained through the Sugawara construction as quadratic forms in the currents: $T(z) = \text{const.} \; : (\mathbf{J}(z))^2 :$. (The colons refer to a specific regularization of this ill-defined product of two currents at coinciding points, and the constant is fixed in any irreducible representation...). For many cases of such an extended chiral algebra, the representation theory has been developed (maybe in less detail for W algebras, even less for other, more exotic, extended algebras ...).

1.3. Elements of Representation theory of the Virasoro algebra

The representations that we shall consider are the *highest weight representations (or Verma modules).* They are parametrized by a real number h: the Verma module $\mathfrak{V}_h$ is generated from a highest weight (h.w.) vector denoted $|h\rangle$ satisfying

$$L_0|h\rangle = h|h\rangle \qquad\qquad L_n|h\rangle = 0, \quad \forall n \in \mathbb{N} \tag{1.10}$$

by the action of the L_{-n}, $n > 0$

$$\mathfrak{V}_h = \mathrm{Span}\{L_{-p_1}L_{-p_2}\cdots L_{-p_r}|h\rangle\}, \quad 1 \le p_1 \le p_2 \le \cdots \le p_r\,. \tag{1.11}$$

Such a representation has the important property of being graded for the action of the Virasoro generator L_0. This means that the spectrum of L_0 in $\mathfrak{V}_h$ is of the form $\{h, h+1, h+2, \cdots\}$. The subspace of eigenvalue $h+N$ is called the eigenspace of level N.

Whether this module is or is not irreducible is the object of a theorem (Kac, Feigin-Fuchs)[2]

Theorem *Let $c = 1 - 6/x(x+1)$, where $x \in \mathbb{C}$. Then $\mathfrak{V}_h$ is reducible iff there exist two positive integers r and s such that*

$$h = h_{rs} := \frac{\left((r(x+1) - sx\right)^2 - 1}{4x(x+1)}\,. \tag{1.12}$$

If h takes one of these values, then $\mathfrak{V}_h$ is reducible in the sense that it contains a "singular" vector, i.e. a vector satisfying the axioms (1.10) of a h.w. vector. This vector thus supports itself a h.w. module, which is a submodule of $\mathfrak{V}_h$. Moreover the theorem asserts that this "degeneracy" occurs at level $r.s$. In fact $\mathfrak{V}_h$ may contain several such submodules, with a non trivial intersection: this is what happens if the parameter x is rational. One constructs the irreducible representation $\mathcal{V}_h$ by quotienting out this (or these) submodule(s) of $\mathfrak{V}_h$.

Assume that the parameter x in (1.12) is of the form $p'/(p-p')$, with p, p' two coprime integers. Show that $h = h_{r\,s} = h_{p'-r\,p-s}$ so that there are degeneracies at the two levels $r.s$ and $(p'-r).(p-s)$ and thus two distinct submodules in $\mathfrak{V}_h$.

Highest weight representations of other chiral algebras may also be constructed. For example, let us sketch the results for an affine algebra $\hat{\mathfrak{g}}$ associated with a simple algebra $\mathfrak{g}$ [3]. Let $\alpha_1, \cdots, \alpha_r$ and θ be the ordinary simple roots and the highest root of the finite algebra $\mathfrak{g}$, $\alpha_i^\vee = 2\alpha_i/(\alpha_i,\alpha_i)$ the corresponding coroots; θ is normalised by $(\theta,\theta) = 2$. Then the so-called *integrable* representations of the algebra $\hat{\mathfrak{g}}$ are labelled by a pair $(\bar{\lambda}, k)$

where k is a non negative integer (the "level" of the representation) and where $\bar\lambda \in P$, (P the weight lattice of $\mathfrak{g}$), is subject to the inequalities

$$(\bar\lambda, \alpha_i^\vee) \in \mathbb{N}, \quad i = 1, \cdots, r \ , \qquad (\bar\lambda, \theta) \le k \ . \tag{1.13}$$

Many formulae simplify if expressed in terms of the shifted weight $\lambda = \bar\lambda + \rho$, with ρ the Weyl vector $\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$. In the case of $\mathfrak{g} = sl(N)$, $r = N-1$, these formulae reduce to $\theta = \sum_1^{N-1} \alpha_i$, the shifted weight $\lambda = \sum_{i=1}^{N-1} \lambda_i \Lambda_i$, with Λ_i the dominant fundamental weights of $sl(N)$, satisfies the inequalities $\lambda_i \ge 1, \sum_{i=1}^{N-1} \lambda_i \le k + N - 1$. The simplest example is of course that of $\widehat{sl}(2)$ for which the representations are labelled by the pair of integers (λ, k), with $1 \le \lambda \le k+1$, (λ may be thought of as $2j+1$, i.e. the dimension of the corresponding finite-dimensional spin j representation of $sl(2)$).

If $\mathcal{V}_i$ is some representation of a chiral algebra, we shall use the label i^* to denote the complex conjugate representation. It may happen that $\mathcal{V}_{i^*}$ is identical or equivalent to $\mathcal{V}_i$, like for Vir, or $\widehat{sl}(2)$. We shall in general label by $i = 1$ the identity representation; its conformal weight (eigenvalue of L_0 on the highest weight vector) vanishes.

As in the case of Vir discussed above, these representations are graded for the action of the Virasoro generator L_0. The spectrum of L_0 in $\mathcal{V}_i$ is of the form $\{h_i, h_i+1, h_i+2, \cdots\}$, with non-trivial multiplicities $\#_n = \dim$ (subspace of eigenvalue $h+n$). It is thus natural to introduce a generating function of these multiplicities, i.e. a function of a dummy variable q, the *character* of the representation $\mathcal{V}_i$

$$\chi_i(q) = \mathrm{tr}_{\mathcal{V}_i} q^{L_0 - \frac{c}{24}} = q^{h_i - \frac{c}{24}} \sum_{n=0}^{\infty} \#_n q^n \ . \tag{1.14}$$

Show that in the original h.w. Verma module $\mathfrak{V}_h$ of Vir, the character is simply $\chi_h(q) = \frac{q^{h-\frac{c}{24}}}{\prod_1^\infty (1-q^n)}$. Assuming that for $c = 1$, the representations with a singular vector have a conformal weight given by the limit $x \to \infty$ of (1.12), namely $h = \ell^2/4$, $\ell \in \mathbb{N}$, $r = \ell + 1$, $s = 1$, show that $\chi_h(q) = \frac{q^{\frac{\ell^2}{4}}(1-q^{\ell+1})}{\eta(q)}$ with Dedekind's eta function: $\eta(q) = q^{\frac{1}{24}} \prod_1^\infty (1-q^n)$. Another interesting case is for $c < 1$, when the parameter x is rational: one writes

$$c = 1 - \frac{6(p-p')^2}{pp'}, \quad p, p' \in \mathbb{N} \tag{1.15}$$

and one concentrates on the irreducible representations ("irrep") with h of the form

$$h_{rs} = h_{p'-r, p-s} = \frac{(rp - sp')^2 - (p-p')^2}{4pp'}, \qquad 1 \le r \le p'-1, \ 1 \le s \le p-1 \ . \tag{1.16}$$

The character of this irrep reads, with $\lambda := (rp - sp')$, $\lambda' := (rp + sp')$

$$\chi_{rs} = \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} \left(q^{\frac{(2npp'+\lambda)^2}{4pp'}} - q^{\frac{(2npp'+\lambda')^2}{4pp'}} \right) \ . \tag{1.17}$$

In representations of the affine algebra $\hat{\mathfrak{g}}$, we may also consider the characters $\mathrm{tr}\, q^{L_0 - c/24}$. They are called "specialized characters" since they count states according to their L_0 grading only. Non-specialized characters can be introduced, which are sensitive to generators of the Cartan subalgebra $\mathbf{J}_0$

$$\chi(q, \mathbf{u}) = \mathrm{tr}\, q^{L_0 - \frac{c}{24}} e^{2\pi i (\mathbf{u}, \mathbf{J}_0)} . \tag{1.18}$$

For the case of $\widehat{sl}(2)$, and for the representations (λ, k) discussed above

$$\chi_\lambda(q) = \frac{1}{\eta^3(q)} \sum_{p=-\infty}^{\infty} (2(k+2)p + \lambda)\, q^{\frac{(2(k+2)p+\lambda)^2}{4(k+2)}} . \tag{1.19}$$

The expressions of non-specialized characters and for general algebras may be found in [3].

Similar considerations apply to other chiral algebras...

1.4. Modular properties of characters

If the multiplicity $\#_n$ doesn't grow too fast in (1.14), this sum converges for $|q| < 1$: it is thus natural to write $q = \exp 2i\pi\tau$, with τ a complex number in the upper half-plane.

It is a remarkable fact that such functions χ enjoy beautiful transformation properties under the action of the modular group on the variable τ

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}) \quad : \tau \mapsto \frac{a\tau + b}{c\tau + d} \tag{1.20}$$

(i.e. a, b, c, d integers defined up to a global sign, with $ad - bc = 1$).

By definition, Rational Conformal Field Theories (RCFT) are CFTs that are consistently described by a finite set $\mathcal{I}$ of representations $\mathcal{V}_i$, $i \in \mathcal{I}$, of a certain chiral algebra $\mathcal{A}$. Moreover the corresponding characters $\chi_i(q)$ form a finite dimensional unitary representation of the modular group (in fact of its double covering, see below): they transform among themselves linearly (and unitarily) under the action of (1.20). It is well known that the modular group is generated by the two transformations

$$T \; : \; \tau \mapsto \tau + 1 \qquad\qquad S \; : \; \tau \mapsto -\frac{1}{\tau} . \tag{1.21}$$

It is clear from the definition (1.14) that under T

$$\chi_i(q) \to \chi_i(q\, e^{2i\pi}) = e^{2i\pi(h_i - \frac{c}{24})} \chi_i(q) \tag{1.22}$$

and the non-trivial part of the above statement is that, if $\tilde{q} := \exp -\frac{2i\pi}{\tau}$ there exists a unitary $|\mathcal{I}| \times |\mathcal{I}|$ matrix S such that

$$\chi_i(q) = \sum_{j \in \mathcal{I}} S_{ij} \chi_j(\tilde{q}) . \tag{1.23}$$

Moreover the matrix S satisfies $S^T = S$, $S^\dagger = S^{-1}$, $(S_{ij})^* = S_{i^*j} = S_{ij^*}$, $S^2 = C =$ the conjugation matrix defined by $C_{ij} = \delta_{ij^*}$, $S^4 = I$. (2)

1st Example: for the $c < 1$ minimal representations,

$$\mathcal{I} = \{(r,s) \equiv (p'-r, p-s)\ ; \quad 1 \le r \le p'-1,\ 1 \le s \le p-1\}$$

with h_{rs} given in (1.16), the S matrix reads

$$S_{rs,r's'} = \sqrt{\frac{8}{pp'}}(-1)^{(r+s)(r'+s')} \sin \pi r r' \frac{p-p'}{p'}\ \sin \pi s s' \frac{p-p'}{p} \tag{1.24}$$

2nd Example: the $\widehat{sl}(2)$ affine algebra

At level k, we have seen that the integrable representations are labelled by the set $\mathcal{I} = \{1, 2, \cdots, k+1\}$. The conformal weight of representation (λ) reads $h_\lambda = (\lambda^2-1)/4(k+2)$, $\lambda \in \mathcal{I}$. Then one finds

$$S_{\lambda\mu} = \sqrt{\frac{2}{k+2}} \sin \frac{\pi \lambda \mu}{k+2}\ , \quad \lambda, \mu \in \mathcal{I}\ . \tag{1.25}$$

For non-specialised characters, the transformation reads:

$$\chi_\lambda(q, \mathbf{u}) = e^{ik\pi(\mathbf{u},\mathbf{u})/\tau} \sum_{\mu \in \mathcal{I}} S_{\lambda\mu} \chi_\mu(\tilde{q}, \mathbf{u}/\tau)\ . \tag{1.26}$$

The expression for more general affine algebras may be found in [3].

One notes that the S matrix of the minimal case (1.24) is "almost" the tensor product of two matrices of the form (1.25), at two different levels $k = p-2$ and $k' = p'-2$. This would be true for $|p-p'| = 1$, and if one could omit the identification $(r,s) \equiv (p'-r, p-s)$. This is of course not a coincidence but reflects the "coset construction" of $c < 1$ representations of Vir out of the affine algebra $\widehat{sl}(2)$ [4].

1.5. Notion of Fusion Algebra

The last concept of crucial importance for our discussion is that of fusion algebra. Fusion is an operation among representations of chiral algebras of RCFTs, inherited from the operator product algebra of Quantum Field Theory. It looks similar to the usual tensor product of representations, but contrary to the latter, it is consistent with the finiteness of the set $\mathcal{I}$ and it preserves the central elements (instead of adding them). I shall refer to the literature [3,5] for a systematic discussion of this concept, and just introduce a notation $\star$ to denote it and distinguish it from the tensor product. It is natural to decompose the fusion of two representations of a chiral algebra on the irreps, thus defining multiplicities, or "fusion coefficients"

$$\mathcal{V}_i \star \mathcal{V}_j = \oplus_k \mathcal{N}_{ij}{}^k\, \mathcal{V}_k, \qquad \mathcal{N}_{ij}{}^k \in \mathbb{N}\ . \tag{1.27}$$

(2) The fact that $S^2 = C$ rather than $S^2 = I$ as expected from the transformation (1.21) signals that we are dealing with a representation of a double covering of the modular group.

There is a remarkable formula, due to Verlinde [6], expressing these multiplicities in terms of the unitary matrix S:

$$\mathcal{N}_{ij}{}^{k} = \sum_{\ell \in \mathcal{I}} \frac{S_{i\ell}\, S_{j\ell}\, (S_{k\ell})^*}{S_{1\ell}} \tag{1.28}$$

Note that the fact that the r.h.s. of (1.28), computed with the matrices (1.25) or (1.24) yields non negative integers is a priori non-trivial!

This completes our review of the basics of RCFTs. The data c (or k, etc), $\mathcal{V}_i, h_i, i \in \mathcal{I}$, $S_i{}^j$, $\mathcal{N}_{ij}{}^k$, form what I call the "chiral data": they are relative to a "chiral half" of the conformal theory (in the plane), which means they refer only to the holomorphic variables z or to their antiholomorphic counterparts $\bar{z}$, rather than to the pair $(z, \bar{z})$.

Our task is now to use these ingredients to construct physically sensible theories.

2. Modular Invariant Partition Functions

In the plane punctured at the origin, equipped with the coordinate z, or equivalently on the cylinder of perimeter L with the coordinate w, with the conformal mapping from the latter to the former $z = \exp -2\pi i w/L$, a given RCFT is described by a Hilbert space $\mathcal{H}_P$. This Hilbert space is decomposable into a *finite* sum of irreps of **two** copies of the chiral algebra (Vir or else), associated with the holomorphic and anti-holomorphic "sectors" of the theory:

$$\mathcal{H}_P = \oplus N_{j\bar{j}} \mathcal{V}_j \otimes \mathcal{V}_{\bar{j}} \ , \tag{2.1}$$

with (non negative integer) multiplicities $N_{j\bar{j}}$. On the cylinder, it is natural to think of the Hamiltonian as the operator of translation along its axis (the imaginary axis in w), or along any helix, defined by its period τL in the w plane, with $\Im m\, \tau > 0$. If L_{-1}^{cyl} and $\bar{L}_{-1}^{\mathrm{cyl}}$ are the two generators of translation in w and $\bar{w}$, $H^{\mathrm{cyl},\tau} = (\tau L_{-1}^{\mathrm{cyl}} + \bar{\tau} \bar{L}_{-1}^{\mathrm{cyl}})$. Mapped back in the plane using the transformation law of the energy-momentum tensor (1.4), L_{-1}^{cyl} reads

$$L_{-1}^{\mathrm{cyl}} = -\frac{2\pi i}{L}(L_0 - \frac{c}{24}) \tag{2.2}$$

where the term $c/24$ comes from the schwarzian derivative of the exponential mapping. The evolution operator of the system, i.e. the exponential of L times the Hamiltonian is thus

$$e^{-H^{\mathrm{cyl},\tau} L} = e^{2\pi i (\tau (L_0 - \frac{c}{24}) - \bar{\tau}(\bar{L}_0 - \frac{c}{24}))} \ . \tag{2.3}$$

A convenient way to encode the information (2.1) is to look at the partition function of the theory on a torus $\mathcal{T}$. Up to a global dilatation, irrelevant here, a torus may be defined by its modular parameter τ, $\Im m\, \tau > 0$, such that its two periods are 1 and τ. Equivalently, it may be regarded as the quotient of the complex plane by the lattice generated by the two numbers 1 and τ:

$$\mathcal{T} = \mathbb{C}/(\mathbb{Z} \oplus \tau \mathbb{Z}) \ , \tag{2.4}$$

in the sense that points in the complex plane are identified according to $w \sim w' = w + n + m\tau$, $n, m \in \mathbb{Z}$. There is, however, a redundancy in this description of the torus: the modular parameters τ and $M\tau$ describe the same torus, for any modular transformation $M \in PSL(2,\mathbb{Z})$. The partition function of the theory on this torus is just the trace of the evolution operator (2.3), with the trace taking care of the identification of the two ends of the cylinder into a torus

$$Z = \mathrm{tr}_{\mathcal{H}_P} e^{2\pi i[\tau(L_0 - \frac{c}{24}) - \bar\tau(\bar L_0 - \frac{c}{24})]} \ . \tag{2.5}$$

Using (2.1) and the definition (1.14) of characters, this trace may be written as

$$Z = \sum N_{j\bar j}\, \chi_j(q)\chi_{\bar j}(\bar q) \qquad q = e^{2\pi i\tau} \quad \bar q = e^{-2\pi i\bar\tau}. \tag{2.6}$$

Let's stress that in these expressions, $\bar\tau$ is the complex conjugate of τ, and $\bar q$ that of q, and therefore, $Z = \sum N_{j\bar j}\chi_j(q)\big(\chi_{\bar j}(q)\big)^*$ is a *sesquilinear* form in the characters. Finally this partition function must be intrinsically attached to the torus, and thus be invariant under modular transformations. This key observation, together with the expression of Z as a sesquilinear form in the characters, is due to Cardy [7]. As we shall now show, it opens a route to the classification of RCFTs. We have been led indeed to the ...

Classification Problem : *find all possible sesquilinear forms (2.6) with non negative integer coefficients that are modular invariant, and such that* $N_{11} = 1$.

The extra condition $N_{11} = 1$ expresses the unicity of the identity representation (i.e. of the "vacuum"). (3) As explained in the previous section, the finite set of characters of any RCFT, labelled by $\mathcal{I}$, supports a unitary representation of the modular group. This implies that any diagonal combination of characters $Z = \sum_{i\in\mathcal{I}}\chi_i(q)\chi_i(\bar q)$ is modular invariant. Are there other solutions to the problem?

The problem has been completely solved only in a few cases: for the RCFTs with an affine algebra, the $\widehat{sl}(2)$ [10] and $\widehat{sl}(3)$ [11] theories at arbitrary level, plus a host of cases with constraints on the level, e.g. the general $\widehat{sl}(N)$ for $k = 1$ [12], etc; associated coset theories have also been fully classified, including all the minimal $c < 1$ theories, $N = 2$ "minimal" superconformal theories, etc. A good review on the current state of the art is provided by T. Gannon [13].

In the case of CFTs with a current algebra, it is in fact better to look at the same problem of modular invariants after replacing in (2.6) all specialized characters by non-specialized ones,

(3) Notice that the property of modular invariance is just a necessary condition of consistency of the theory. It may be –and it seems to happen– that some modular invariants do not correspond to any consistent CFT. The general conditions to be fulfilled by a CFT to be fully consistent have been spelled out by Sonoda [8] and by Moore and Seiberg [9]. They amount essentially to the consistency (duality equations) of the 4-point function in the plane, and modular invariance (or covariance) of the 0- and 1-point functions on the torus. Nobody has been able, however, to analyse systematically these conditions besides the simplest cases of $sl(2)$-related theories.

v.i.z. $\sum N_{j\bar{j}}\chi_j(q,\mathbf{u})\Big(\chi_{\bar{j}}(q,\mathbf{u})\Big)^*$. Because these non-specialized characters are linearly independent, there is no ambiguity in the determination of the multiplicities $N_{j\bar{j}}$ from Z. This alternative form of the partition function may be seen as resulting from a modification of the energy-momentum tensor $T(z) \to T(z) - \frac{2\pi i}{L}(\mathbf{u}, \mathbf{J}(z)) - \frac{k}{2}\left(\frac{2\pi}{L}\right)^2(\mathbf{u},\mathbf{u})$, see [14].

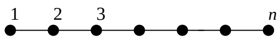
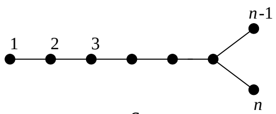
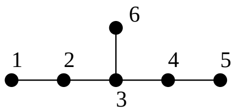
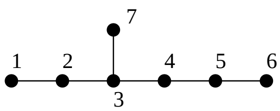
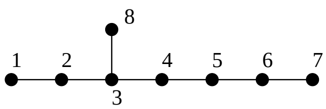
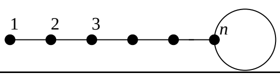
2.1. The $\widehat{sl}(2)$ cases

This was the first case fully solved. With the notations introduced in the previous section for the representations of $\widehat{sl}(2)$, the complete list of modular invariant partition functions is as listed in Table 1. A remarkable feature appears, namely an unexpected connection with ADE Dynkin diagrams. By this I mean that if we concentrate on the *diagonal* terms of these expressions, their labels λ turn out to be the Coxeter *exponents* of these Dynkin diagrams. I recall that for such a diagram, the eigenvalues of its adjacency matrix G are of the form $2\cos\frac{\pi\ell}{h}$, h the Coxeter number, and ℓ is an exponent taking rank(G) (= number of vertices of G) values between 1 and $h-1$, with possible multiplicities. Alternatively, the Cartan matrix $C = 2\mathbb{I} - G$ has eigenvalues $4\sin^2\frac{\pi\ell}{2h}$. These Coxeter number and exponents are listed in Table 2 (do not pay attention for the time being to the last entry denoted T_n). Anticipating a little on the following, let's notice that the off-diagonal terms in the partition functions of Table 1 may also be determined in terms of the data of the ADE diagrams.

Table 1
List of modular invariant partition functions in terms of $SU(2)$ Kac–Moody characters χ_λ.

$k \geq 0$	$\displaystyle\sum_{\lambda=1}^{k+1} \lvert\chi_\lambda\rvert^2$	(A_{k+1})
$k = 4\rho \geq 4$	$\displaystyle\sum_{\lambda\ \mathrm{odd}\ =1}^{2\rho-1} \lvert\chi_\lambda + \chi_{4\rho+2-\lambda}\rvert^2 + 2\lvert\chi_{2\rho+1}\rvert^2$	$(D_{2\rho+2})$
$k = 4\rho - 2 \geq 6$	$\displaystyle\sum_{\lambda\ \mathrm{odd}\ =1}^{4\rho-1} \lvert\chi_\lambda\rvert^2 + \lvert\chi_{2\rho}\rvert^2 + \sum_{\lambda\ \mathrm{even}\ =2}^{2\rho-2} (\chi_\lambda\chi^*_{4\rho-\lambda} + \mathrm{c.c.})$	$(D_{2\rho+1})$
$k = 10$	$\lvert\chi_1+\chi_7\rvert^2 + \lvert\chi_4+\chi_8\rvert^2 + \lvert\chi_5+\chi_{11}\rvert^2$	(E_6)
$k = 16$	$\lvert\chi_1+\chi_{17}\rvert^2 + \lvert\chi_5+\chi_{13}\rvert^2 + \lvert\chi_7+\chi_{11}\rvert^2 + \lvert\chi_9\rvert^2$ $+[(\chi_3+\chi_{15})\chi_9^* + \mathrm{c.c.}]$	(E_7)
$k = 28$	$\lvert\chi_1+\chi_{11}+\chi_{19}+\chi_{29}\rvert^2 + \lvert\chi_7+\chi_{13}+\chi_{17}+\chi_{23}\rvert^2$	(E_8)

Table 2: The graphs A-D-E-T, their Coxeter number and their Coxeter exponents

		h	exponents
A_n		$n+1$	$1, 2, \cdots, n$
$D_{\ell+2}$		$2(\ell+1)$	$1, 3, \cdots, 2\ell+1, \ell+1$
E_6		12	$1, 4, 5, 7, 8, 11$
E_7		18	$1, 5, 7, 9, 11, 13, 17$
E_8		30	$1, 7, 11, 13, 17, 19, 23, 29$
T_n		$2n+1$	$1, 3, 5, \cdots, 2n-1$

2.2. On ADE classifications

The content of this section is not essential for what follows (except point (vii) below). The subject is so intriguing and so fascinating, however, that I cannot resist presenting it.

It is well known that there are many mathematical objects that fall in an ADE classification [15]. The list includes

(i) simple simply-laced Lie algebras, i.e. with roots of equal length [16];
(ii) finite reflection groups of cristallographic and of simply-laced type [17];
(iii) finite subgroups of $SO(3)$ or of $SU(2)$, (or the associated platonic solids);
(iv) Kleinian singularities [18];
(v) "simple" singularities, i.e. with no modulus [19];
(vi) finite type quivers [20];
(vii) symmetric matrices with eigenvalues between -2 and $+2$;
(viii) algebraic solutions to the hypergeometric equation ([21], p 385);
(ix) subfactors of finite index [22], see also D. Evans' lectures at this school;
and presumably others...

To elaborate a little on these various items:

(i) Simple Lie algebras have been classified by Killing and Cartan; restricting ourselves to the simply laced ones, i.e. with roots of equal length, leaves us with the ADE cases.
(ii) Reflection groups are groups generated by reflections in hyperplanes orthogonal to vectors $\{\alpha_a\}$ in the Euclidean space $\mathbb{R}^n$ called roots : $S_a\ :\ x \mapsto x - 2\alpha_a\,(\alpha_a, x)/(\alpha_a, \alpha_a)$. The group is of finite order iff the bilinear form (α_a, α_b) is positive definite. This leads to a list $A_n, B_n \equiv C_n, D_n, E_6, E_7, E_8, F_4, G_2, H_3, H_4, I_2(k)$, where the subscript gives the space dimension n [17]. If moreover, the condition that the root system is crystallographic is imposed, (i.e. that for all pairs of roots α, β, $2(\alpha,\beta)/(\beta,\beta) \in \mathbb{Z}$),

only the cases A to G_2 are left: they are of course the Weyl groups of the Lie algebras of (i). Imposing also the condition of simple lacedness leaves only ADE.

(iii) Finite subgroups of $SU(2)$ form two infinite series and three exceptional cases: the cyclic groups $\mathcal{C}_n$, the binary dihedral groups $\mathcal{D}_n$, the binary tetrahedral group $\mathcal{T}$, the binary octahedral group $\mathcal{O}$ and the binary icosahedral group $\mathcal{I}$. It is natural to label them by ADE as we shall see (Table 3). A related classification is that of the 5 regular solids in three-dimensional Euclidean space: this may be the oldest ADE classification, since it goes back to the school of Plato; strictly speaking, only the exceptional cases E_6, E_7, E_8 appear there, since E_6 is associated with the group of the tetrahedron, E_7 with the group of the octahedron or of the cube, and E_8 with the group of the dodecahedron or of the icosahedron. The cyclic and dihedral groups may be thought of respectively as the rotation invariance group of a pyramid and of a prismus of base a regular n-gon, but those are not regular platonic solids.

(iv) Kleinian singularities: Let Γ be a finite subgroup of $SU(2)$. It acts on $(u,v) \in \mathbb{C}^2$. The algebra of Γ-invariant polynomials in u,v is generated by three polynomials X,Y,Z subject to one relation $W(X,Y,Z)=0$. The quotient variety $\mathbb{C}^2/\Gamma$ is parametrized by these polynomials X,Y,Z and is thus embedded into the hypersurface $W(x,y,z)=0$, $x,y,z \in \mathbb{C}^3$. This variety $\mathcal{S}$ is singular at the origin [18], see Table 3.

(v) Simple singularities, i.e. polynomials $W(x_1,x_2,\cdots,x_p)$ with $\partial W/\partial x_j|_0 = 0$ for all $i=1,\cdots,p$, and with no modulus (up to regular changes of the x), are also given by the same Kleinian polynomials W (up to the addition or subtraction of quadratic terms) [19].

(vi) Quivers are oriented graphs, with vertices a and edges e. A representation of a quiver is the assignment to each vertex a of a non-negative integer d_a and of a vector space $V_a \equiv \mathbb{C}^{d_a}$ and to each edge $e=(a \to b)$ of a linear map $f_e: \ V_a \mapsto V_b$. Two such representations (V_a, f_e) and (W_a, g_e) are equivalent if there are linear maps $\varphi_a: V_a \mapsto W_a$ such that $\varphi_b \circ f_e = g_e \circ \varphi_a$ for all edges $e=(a,b)$. One proves that quivers of finite type, i.e. such that they admit only a finite number of inequivalent indecomposable representations, are ADE diagrams ! [20],[18].

(vii) Symmetric matrices with non negative integer entries and eigenvalues between -2 and 2 are the adjacency matrices of the $ADET$ graphs of Table 2. The "tadpoles" $T_n = A_{2n}/\mathbb{Z}_2$ may be ruled out if the condition of 2-colourability (or "bipartiteness") is imposed [23].

Γ	$\mathcal{C}_n$	$\mathcal{D}_n$	$\mathcal{T}$	$\mathcal{O}$	$\bar{\mathcal{I}}$
$\|\Gamma\|$	n	$4n$	24	48	120
W	$X^n - YZ$	$X^{n+1}+XY^2+Z^2$	$X^4+Y^3+Z^2$	$X^3+XY^3+Z^2$	$X^5+Y^3+Z^2$
G	A_{n-1}	D_{n+2}	E_6	E_7	E_8

Table 3: Finite subgroups of $SU(2)$, their orders, the Kleinian singularity and the associated Dynkin diagram.

The most obvious manifestation of the ADE classification of these objects is provided by the Dynkin diagram and its exponents. In cases (i) and (ii) the Dynkin diagram encodes the geometry of the root system $\{\alpha_a\}$: $(\alpha_a, \alpha_b) = C_{ab}$ the Cartan matrix $= 2\mathbb{I} - G_{ab}$, G

the adjacency matrix, while the exponents shifted by 1 give the degrees of the invariant polynomials. Also in case (ii), the product over all the simple roots of the reflections S_a defines the Coxeter element, unique up to conjugation, whose eigenvalues are $\exp 2i\pi m_i/h$, m_i running over the exponents. In case (iii), as found by McKay [24], things are subtle and beautiful: the corresponding *affine* Dynkin diagrams of $\hat{A}$-$\hat{D}$-$\hat{E}$ type describe the decomposition into irreducible representations of the tensor products of the representations of Γ by a two dimensional representation. By removing the vertex corresponding to the identity representation, one recovers the standard ADE Dynkin diagrams in accordance with Table 3. See Appendix A for more elements on the McKay correspondence. In the case (iv) of Kleinian singularities, one considers the *resolution* $\tilde{\mathcal{S}}$ of the singular surface $\mathcal{S}$: this is a smooth variety with a projection $\pi : \tilde{\mathcal{S}} \to \mathcal{S}$ which is one-to-one everywhere except above the singularity at 0: one proves that the *exceptional divisor* $\pi^{-1}(0)$ is a connected union of spheres, $\pi^{-1}(0) = C_1 \cup \cdots \cup C_r$, $C_i \cong \mathbb{P}^1\mathbb{C}$. The Dynkin diagram of ADE type listed on the last line of Table 3 (or more precisely the negative of its Cartan matrix) describes the intersection form of these components C_i. In the case (v) of a simple singularity, one may consider its deformation $W = \epsilon$ and look at the intersection of the homology cycles of its level set $\{x \in \mathbb{C}_p,\ |x| < \delta \mid W(x) = \epsilon\}$, or at their monodromy as ϵ circles around the origin: the intersection is again encoded in the Dynkin diagram, while the monodromy is given by the Coxeter element of the associated Coxeter group. Shifted by -1 the exponents give the degrees of the homogeneous polynomials of the local ring $\mathbb{C}[x_1, \cdots, x_p]/(\partial_{x_i} W)$ of the simple singularity [19] etc, etc.

This is just a sample of all the fascinating properties and crossrelations between these problems.

In many cases, the classification follows from the spectral condition (vii). In some others, however, the key point is the determination of triplets of integers (p, q, r) such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$. (Prove that the ADE list below includes all the solutions except $(p = 1, q \neq r)$.) Note also that for the D and E cases, these integers give the length of the three branches of the Dynkin diagram counted from the vertex of valency 3. (The A entry may look slightly artificial at this stage).

G	A_{2n-1}	D_{n+2}	E_6	E_7	E_8
(p, q, r)	$(1, n, n)$	$(2, 2, n)$	$(2, 3, 3)$	$(2, 3, 4)$	$(2, 3, 5)$

The integers (p, q, r) also appear in the definition of the binary polyhedral groups by generators and relations : $R^p = S^q = T^q = RST$, $(RST)^2 = 1$. The relationship between these two occurrences of (p, q, r) is a nice manifestation of the "dual McKay correspondence" of [25].

To the above list, we have now added one more item: the $\widehat{sl}(2)$ modular invariant partition functions. A natural question is: does this new item connect to any previously known case? If it is so, this may give us a hint about what may be expected in other cases. For example, is there some spectral property on matrices that would be related to the modular invariants of $\widehat{sl}(3)$ type? Or would the subgroups of $SU(3)$ be of relevance? Or a certain class of singularities beyond the "simple" ones?

I think it is fair to say that the question has not yet received a clear answer. We do see connections between existing ADE classifications and some consequences of the ADE

classification of $\widehat{sl}(2)$ modular invariants, but we do not see how they extend to higher rank cases. Or at least, in no direct and systematic way ... This is what we shall show by discussing next the case of $\widehat{sl}(3)$ theories (sect. 2.3). Then in section 3, we shall see that the study of CFT in the presence of boundaries ("BCFT") gives us some new insight in these questions.

In the $\widehat{sl}(2)$ case, there are two related classes of CFTs for which another ADE classification appears from another standpoint: the $c < 1$ (unitary) minimal models, and the "simple" $\mathcal{N} = 2$ superconformal field theories or their topological cousins [26]. Both may be obtained by the coset construction from the $\widehat{sl}(2)$ models, and inherit from them a variant of the ADE classification. In the former case, it is known that $c < 1$ minimal models admit a lattice integrable realisation [27]: the configuration space of these lattice models is the space of paths on a graph, and demanding that this space supports a representation of the Temperley-Lieb algebra (a quantum deformation of the symmetric group algebra and a known way to achieve integrability) and that the model is critical (in the sense of statistical mechanics) forces us to restrict to graphs with eigenvalues of their adjacency matrix between -2 and 2, hence of ADE type. On the other hand, the $\mathcal{N} = 2$ superCFTs have been argued to admit a description of their "chiral sector" by a Landau-Ginsburg superpotential which must be one of the simple singularities described above, whence again of type ADE [28]. Thus in these two cases, we have an alternative way to see why and how ADE appears. Unfortunately, these alternative standpoints are of little help in the higher rank cases: the class of graphs supporting a representation of the Hecke algebra associated with $sl(N)$, $N > 3$, is not yet known, (for $N = 3$, see [29,30,31,32]) and the $\mathcal{N} = 2$ theories related to higher $sl(N)$ are not all described by a Landau-Ginsburg potential.

There is a finer subdivision of items classified by ADE into two classes: those classified by $A, D_{2\ell}, E_6$, or E_8 and those classified by $D_{2\ell+1}$ or E_7. The distinction appears in many cases when one looks at positivity properties of some numbers, coefficients, etc. For example, in the list of Table 1, the modular invariants of the first class may be written as sums of squares of linear combinations of χ with non negative coefficients. I am not sure that the relevant positivity property has been identified in all cases listed above (see [33] for a discussion of some aspects of this issue). Ocneanu has shown that Dynkin diagrams of the first subclass admit a "flat connection" [34]. Another manifestation of the distinction is the existence or non-existence of a fusion-like algebra, called the Ocneanu-Pasquier algebra, attached to the Dynkin diagram [35,36,33]. From the point of view of CFT, this apparently innocent looking distinction reveals a different structure of the theory. The block-diagonal modular invariant partition functions classified by $A, D_{2\ell}, E_6, E_8$ may be regarded as diagonal in terms of characters of some extended chiral algebra; the others are obtained from the latter by some twisting procedure [9,37]. All these considerations extend beyond the case of $\widehat{sl}(2)$ theories.

2.3. The $sl(3)$ case, the associated graphs

Let us turn to the case of $\widehat{sl}(3)$ for which complete results are now available. According to what was said above on the affine algebras $\widehat{sl}(N)$ at a given level k, each integrable representation of $\widehat{sl}(3)_k$ is labelled by a weight $\lambda = \lambda_1\Lambda_1 + \lambda_2\Lambda_2$ subject to inequalities $\lambda_i \geq 1, \lambda_1 + \lambda_2 \leq k + 2$. With these notations, the complete list of modular invariants is given in Table 4 [11]. It includes four infinite series and six exceptional cases. In the same way as the modular invariants of $\widehat{sl}(2)$ were associated with Dynkin diagrams, with the

exponents of the latter giving the diagonal terms of the former, we find here the

Fact *With each modular invariant Z of $\widehat{sl}(3)$ one may associate (at least) one graph, whose spectrum is encoded in the diagonal terms of Z. More precisely, the adjacency matrix of this graph has eigenvalues of the form: $S_{(2,1),\lambda}/S_{(1,1),\lambda}$, with a multiplicity equal to the diagonal element $N_{\lambda,\lambda}$ of Z.*

This fact was originally proposed as an Ansatz and graphs were found by empirical methods [29]. Later, more systematic techniques to determine the graph were developed in a variety of cases [38], but the physical interpretation of the graph itself remained unclear until recently, when it took a new perspective in the light of boundary conformal field theory (Lecture 3). On a more abstract level, this also inspired developments by Ocneanu, by Xu and by Böckenhauer, Evans and Kawahigashi [39,31,40]. The list of $\widehat{sl}(3)$ graphs displayed in Fig. 1 and 2 were recapitulated in a recent work with Behrend, Pearce and Petkova [14].

Let us discuss briefly the properties of these graphs (Fig. 1 and 2).

$\star$ Most of them turn out to be 3-colourable: a colour, black (b), gray (g) or white (w), may be assigned to each vertex, and it is understood that the oriented edges go as: $b \to g \to w \to b$. Some of the graphs, however, are not 3-colourable, and then the orientation of the edges has been explicitly displayed whenever necessary, while the absence of arrow in these cases means that the edge carries both orientations.

$\star$ All the graphs listed in Fig. 1 and 2 satisfy the spectral property stated above, but many others also do, which are not associated with any modular invariant [29].

$\star$ There are a few cases for which two graphs are associated with the same modular invariant, for example $\mathcal{D}^{(6)}$ and $\mathcal{D}^{(6)*}$, or $\mathcal{E}_1^{(12)}$ and $\mathcal{E}_2^{(12)}$. The graph $\mathcal{E}_3^{(12)}$ which is isospectral with the two latter and was believed so far to be also associated with the same modular invariant, is discarded by A. Ocneanu on the basis that it does not support a system of "triangular cells" [32], i.e., presumably, that the corresponding CFT has no consistent Operator Algebra. It is thus marked with a question mark on Fig. 2.

$\star$ Many of these graphs may be obtained by the following procedure, inspired by the McKay correspondence for $SU(2)$. Given a finite subgroup Γ of $SU(3)$, the decomposition into irreducibles (b) of the tensor product of each irreducible representation (a) of Γ by the restriction to Γ of the defining three-dimensional representation of $SU(3)$ yields a matrix $\hat{G}$: $(a) \otimes (3) = \oplus_b \hat{G}_{ab}(b)$. Some appropriate truncation of the graph of $\hat{G}$ may then yield a graph adequate for our problem. Contrary to the case of $SU(2)$, however, things are neither systematic –which vertices/edges have to be deleted is not clear a priori– nor exhaustive: some graphs like $\mathcal{E}^{(24)}$ in Fig. 2 are not reached by this procedure.

$\star$ It may be interesting, in view of point (v) of the ADE list above, to note that some of these graphs enable one to construct a reflection group associated with a singularity [41],[42].

$\star$ For a proof that the list of graphs is complete from the subfactor perspective see [32]. See also the discussion in [31],[40], where several of these graphs have been reproduced.

Of course, there is nothing special with $\widehat{sl}(3)$ at this stage, and everything could be repeated for higher rank, except that no complete list of modular invariants nor of graphs is known in these cases. Also, for $\widehat{sl}(N)$, it is in fact a collection of $N-1$ graphs, labelled by the fundamental representations of $SU(N)$, which must be provided. Complex conjugate representations give graphs with all orientations reversed, and the first $[N/2]$ are thus sufficient.

2.4. General case

In general, we expect that a graph (or a collection of graphs) will be associated with any "rational" conformal field theory, (see section 1.4), with the spectrum of its adjacency

Table 5. List of $\widehat{sl}(3)_k$ modular invariants and associated graphs.

The superscript n on the graph must equal $k+3$.

$$(\mathcal{A}^{(n)}) \qquad Z = \sum_{\lambda \in P_{++}^{(n)}} |\chi_\lambda|^2$$

$$(\mathcal{A}^{(n)*}) \qquad Z = \sum_{\lambda \in P_{++}^{(n)}} \chi_\lambda {\chi_{\lambda^*}}^*$$

$$(\mathcal{D}^{(n)}) \qquad Z = \begin{cases} \frac{1}{3}\sum_{\lambda \in Q\cap P_{++}^{(n)}} |\chi_\lambda + \chi_{\sigma\lambda} + \chi_{\sigma^2\lambda}|^2 & \text{if 3 divides } n \\ \sum_{\lambda \in Q\cap P_{++}^{(n)}} |\chi_\lambda|^2 + \sum_{\lambda \in P_{++}^{(n)} \setminus Q} \chi_\lambda \chi^*_{\sigma^{n\tau}\lambda} & \text{if 3 does not divide } n \end{cases}$$

$$(\mathcal{D}^{(n)*}) \qquad Z = \begin{cases} \frac{1}{3}\sum_{\lambda \in Q\cap P_{++}^{(n)}} \left(\sum_{\ell=0}^{2} \chi_{\sigma^\ell\lambda}\right)\left(\sum_{\ell=0}^{2} \chi_{\sigma^\ell\lambda^*}\right)^* & \text{if 3 divides } n \\ \sum_{\lambda \in Q\cap P_{++}^{(n)}} \chi_\lambda \chi_{\lambda^*} + \sum_{\lambda \in P_{++}^{(n)} \setminus Q} \chi_\lambda \chi^*_{\sigma^{-n\tau}\lambda^*} & \text{if 3 does not divide } n \end{cases}$$

$$(\mathcal{E}^{(8)}) \qquad Z = |\chi_{(1,1)} + \chi_{(3,3)}|^2 + |\chi_{(3,2)} + \chi_{(1,6)}|^2 + |\chi_{(2,3)} + \chi_{(6,1)}|^2 + |\chi_{(4,1)} + \chi_{(1,4)}|^2 + |\chi_{(1,3)} + \chi_{(4,3)}|^2 + |\chi_{(3,1)} + \chi_{(3,4)}|^2$$

$$(\mathcal{E}^{(8)*}) \qquad Z = |\chi_{(1,1)} + \chi_{(3,3)}|^2 + \big(\chi_{(3,2)} + \chi_{(1,6)}\big)\big(\chi_{(2,3)} + \chi_{(6,1)}\big)^* + \text{c.c.} + |\chi_{(4,1)} + \chi_{(1,4)}|^2 + \big(\chi_{(1,3)} + \chi_{(4,3)}\big)\big(\chi_{(3,1)} + \chi_{(3,4)}\big)^* + \text{c.c.}$$

$$(\mathcal{E}_i^{(12)}, i = 1,2 \text{ (and 3?)}) \qquad Z = |\chi_{(1,1)} + \chi_{(10,1)} + \chi_{(1,10)} + \chi_{(5,5)} + \chi_{(5,2)} + \chi_{(2,5)}|^2 + 2|\chi_{(3,3)} + \chi_{(3,6)} + \chi_{(6,3)}|^2$$

$$(\mathcal{E}_5^{(12)}) \qquad Z = |\chi_{(1,1)} + \chi_{(10,1)} + \chi_{(1,10)}|^2 + |\chi_{(3,3)} + \chi_{(3,6)} + \chi_{(6,3)}|^2 + |\chi_{(5,5)} + \chi_{(5,2)} + \chi_{(2,5)}|^2 + |\chi_{(4,7)} + \chi_{(7,1)} + \chi_{(1,4)}|^2$$
$$+ |\chi_{(7,4)} + \chi_{(1,7)} + \chi_{(4,1)}|^2 + 2|\chi_{(4,4)}|^2 + \big(\chi_{(2,2)} + \chi_{(8,2)} + \chi_{(2,8)}\big)\chi^*_{(4,4)} + \text{c.c.}$$

$$(\mathcal{E}_4^{(12)*}) \qquad Z = |\chi_{(1,1)} + \chi_{(10,1)} + \chi_{(1,10)}|^2 + |\chi_{(3,3)} + \chi_{(3,6)} + \chi_{(6,3)}|^2 + |\chi_{(5,5)} + \chi_{(5,2)} + \chi_{(2,5)}|^2 + 2|\chi_{(4,4)}|^2$$
$$+ \big(\chi_{(4,7)} + \chi_{(7,1)} + \chi_{(1,4)}\big)\big(\chi_{(7,4)} + \chi_{(1,7)} + \chi_{(4,1)}\big)^* + \text{c.c.} + \big(\chi_{(2,2)} + \chi_{(8,2)} + \chi_{(2,8)}\big)\chi^*_{(4,4)} + \text{c.c.}$$

$$(\mathcal{E}^{(24)}) \qquad Z = |\chi_{(1,1)} + \chi_{(22,1)} + \chi_{(1,22)} + \chi_{(5,5)} + \chi_{(5,14)} + \chi_{(14,5)} + \chi_{(11,11)} + \chi_{(11,2)} + \chi_{(2,11)} + \chi_{(7,7)} + \chi_{(7,10)} + \chi_{(10,7)}|^2$$
$$+ |\chi_{(7,1)} + \chi_{(16,7)} + \chi_{(1,16)} + \chi_{(1,7)} + \chi_{(7,16)} + \chi_{(16,1)} + \chi_{(5,8)} + \chi_{(11,5)} + \chi_{(8,11)} + \chi_{(8,5)} + \chi_{(5,11)} + \chi_{(11,8)}|^2$$

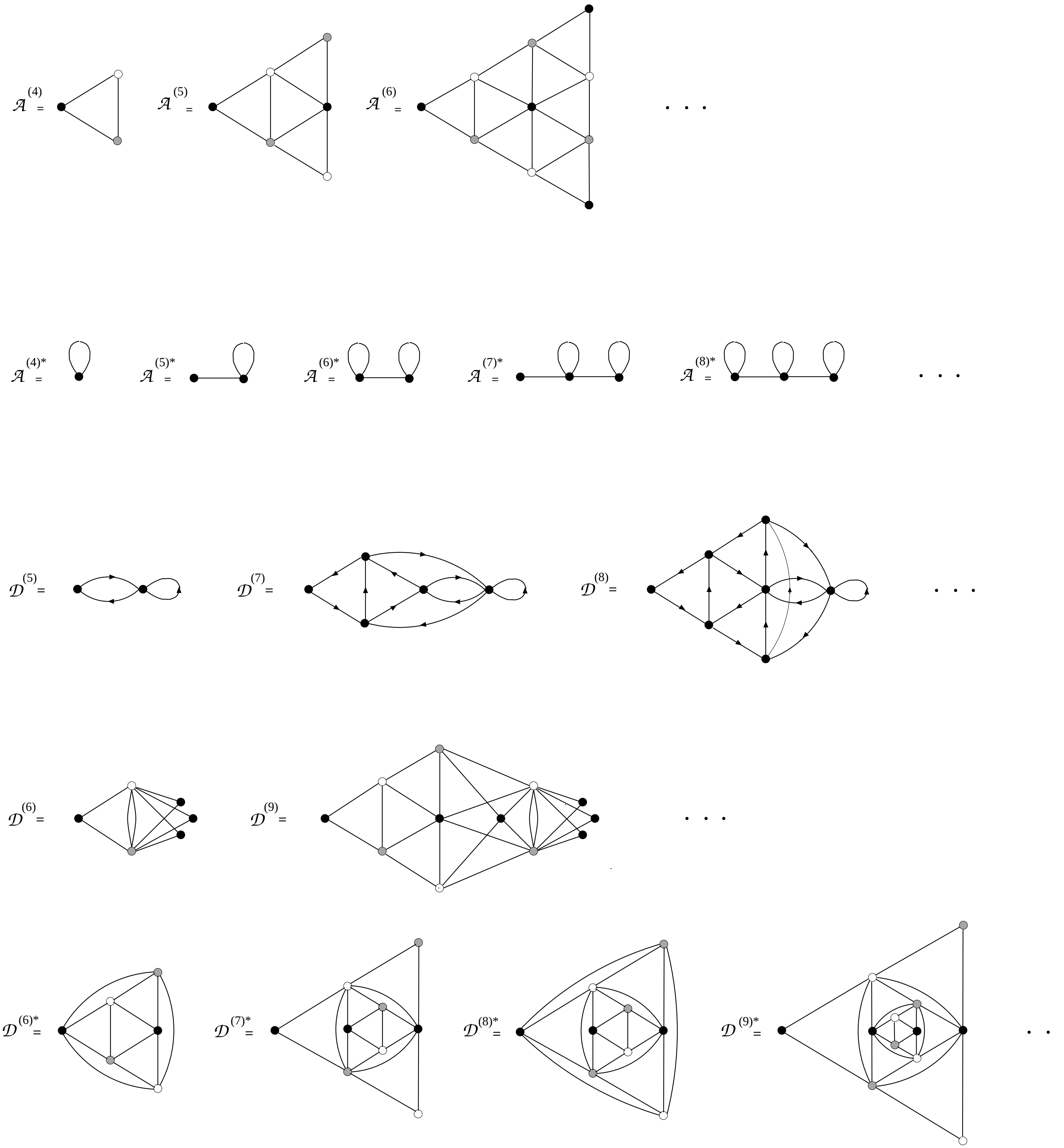


Fig. 1: The graphs of $\widehat{sl}(3)$

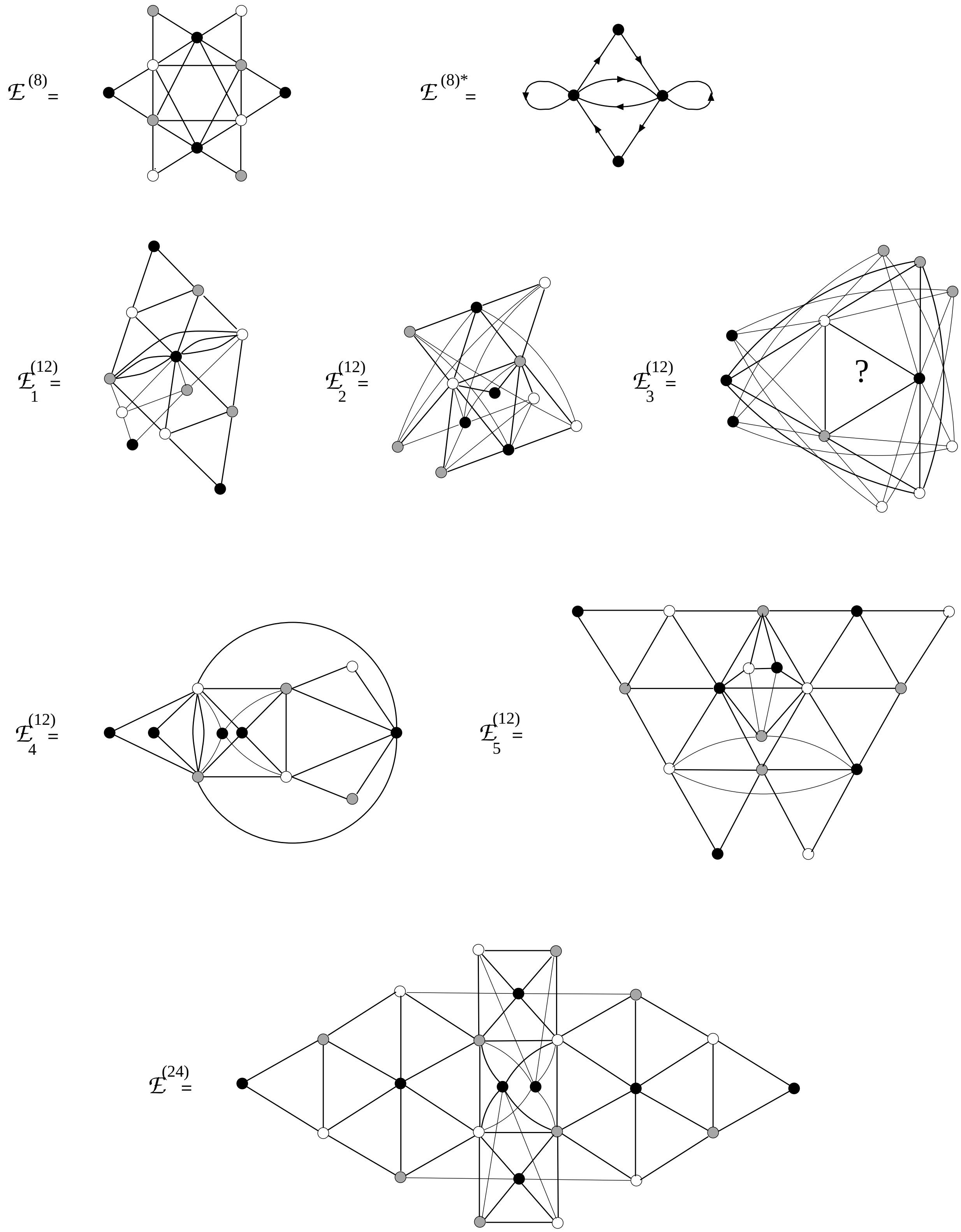


Fig. 2: The graphs of $\widehat{sl}(3)$ (continued)

matrix determined by the diagonal terms of the partition function. Among the pairs $(j, \bar{j})$ appearing in Z, a special role will therefore be played by the diagonal subset

$$\mathcal{E} = \{j | j = \bar{j}, N_{jj} \neq 0\} \ , \tag{2.7}$$

the elements of which, the "exponents" of the theory, will be counted with the multiplicity N_{jj}. Note that $\mathcal{E}$ is stable under conjugation: j and j^* occur with the same multiplicity. The justification of this association of one (or several) graph(s) with a CFT will appear in the next section.

3. Boundary Conformal Field Theory

3.1. RCFT in the half-plane, boundary conditions and operator content

We now turn to the study of RCFTs in a half-plane. There are several physical reasons to look at this problem, –critical systems in the presence of a boundary, open strings and generalized D-branes, one-dimensional electronic systems with "quantum impurities" etc. Here we shall only look at the new information and perspective that this situation gives us in the classification problem of RCFT.

In a half-plane, the admissible diffeomorphisms must respect the boundary, taken as the real axis: thus only real analytic changes of coordinates, satisfying $\epsilon(z) = \bar{\epsilon}(\bar{z})$ for $z = \bar{z}$ real, are allowed. The energy momentum itself has this property:

$$T(z) = \bar{T}(\bar{z})|_{\text{real axis}} \ , \tag{3.1}$$

which expresses simply the absence of momentum flow across the boundary and which enables one to extend the definition of T to the lower half-plane by $T(z) := \bar{T}(z)$ for $\Im m\, z < 0$. There is thus only one copy of the Virasoro algebra $L_n = \bar{L}_n$. This continuity equation (3.1) on T extends to more general chiral algebras and their generators, at the price however of some complication. In general, the continuity equation on generators of the chiral algebra involves some automorphism of that algebra:

$$W(z) = \Omega \bar{W}(\bar{z})|_{\text{real axis}} \tag{3.2}$$

(see [14] and further references therein).

The half-plane, punctured at the origin, (which introduces a distinction between the two halves of the real axis), may also be conformally mapped on an infinite horizontal strip of width L by $w = \frac{L}{\pi} \log z$. Boundary conditions, loosely specified at this stage by labels a and b, are assigned to fields on the two boundaries z real > 0, < 0 or $\Im m\, w = 0, L$. For given boundary conditions on the generators of the algebra and on the other fields of the theory, i.e. for given automorphisms Ω and given a, b, we may again use a description of the system by a Hilbert space of states $\mathcal{H}_{ba}$ (we drop the dependence on Ω for simplicity). On the half-plane or on the finite-width strip, only **one copy** of the Virasoro algebra, or of the chiral algebra $\mathcal{A}$ under consideration, acts on $\mathcal{H}_{ba}$, and this space decomposes on representations of Vir or $\mathcal{A}$ according to

$$\mathcal{H}_{ba} = \oplus n_{ib}{}^{a} \mathcal{V}_i \ , \tag{3.3}$$

with a new set of multiplicities ${n_{ib}}^a \in \mathbb{N}$. The natural Hamiltonian on the strip is the translation operator in $\Re e\, w$, hence, mapped back in the half-plane

$$H_{b|a} = \frac{2\pi}{L}\left(L_0 - \frac{c}{24}\right) \ . \tag{3.4}$$

To summarize, in order to fully specify the operator content of the theory in various configurations, we need not only determine the multiplicities "in the bulk" $N_{j\bar{j}}$ of (2.1), but also the possible boundary conditions a, b on a half-plane and the associated multiplicities ${n_{ib}}^a$. This will be our task in the following, and as we shall see, a surprising fact is that the latter have some bearing on the former.

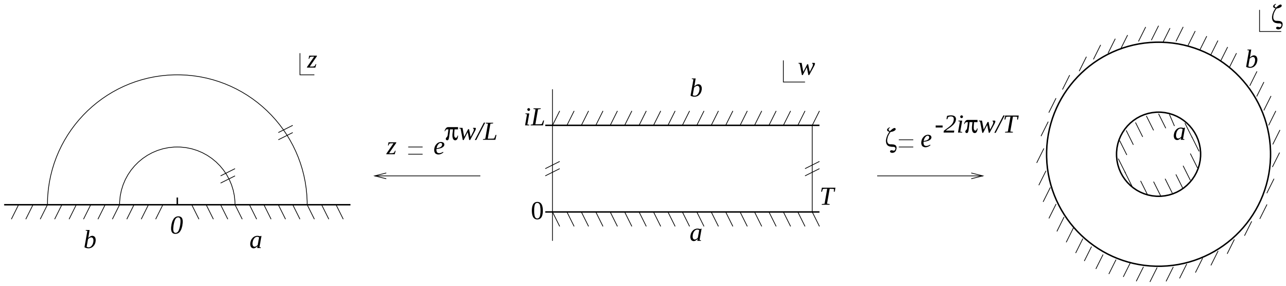


Fig. 3: The same domain seen in different coordinates: a semi-circular annulus, with the two half-circles identified, a rectangular domain with two opposite sides identified, and a circular annulus.

3.2. Boundary states

In the same way that we found useful to chop a finite segment of the infinite cylinder and identify its ends to make a torus, it is suggested to consider a finite segment of the strip – or a semi-annular domain in the half-plane– and identify its edges, thus making a cylinder. This cylinder can be mapped back into an annular domain in the plane, with open boundaries. More explicitly, consider the segment $0 \le \Re e\, w \le T$ of the strip –i.e. the semi-annular domain in the upper half-plane comprised between the semi-circles of radii 1 and $e^{\pi T/L}$, the latter being identified. It may be conformally mapped into an annulus in the complex plane by $\zeta = \exp(-2i\pi w/T)$, of radii 1 and $e^{2\pi L/T}$, see Fig. 3. By working out the effect of this change of coordinates on the energy-momentum T, using (1.4), one finds that (3.1) implies

$$\zeta^2 T(\zeta) = \bar{\zeta}^2 \overline{T}(\bar{\zeta}) \quad \text{for } |\zeta| = 1,\ e^{2\pi\frac{L}{T}} \ . \tag{3.5}$$

Exercise : assuming that W transforms as a primary field of conformal weight $(h_W, 0)$, find the corresponding condition on $W(\zeta)$.

After radial quantization, this translates into a condition on *boundary states* $|a\rangle$, $|b\rangle \in \mathcal{H}_P$ which describe the system on these two boundaries.

$$(L_n - \bar{L}_{-n})|a\rangle = 0 \tag{3.6}$$

(and likewise for $|b\rangle$). Analogously $(W_n - (-1)^{h_W}\Omega(\bar{W}_{-n}))|a\rangle = 0$, with h_W = spin of W.

We shall now look for a basis of states, solutions of this linear system of boundary conditions. One may look for solutions of these equations in each $\mathcal{V}_j \otimes \mathcal{V}_{\bar{j}} \subset \mathcal{H}_P$, since these spaces are invariant under the action of the two copies of Vir or of the chiral algebra $\mathcal{A}$. Consider only for simplicity the case of the Virasoro generators.

Lemma *There is an independent "Ishibashi state" $|j\rangle\rangle$, solution of (3.6), for each $j = \bar{j}$, i.e. $j \in \mathcal{E}$, the set of exponents.*

Proof (G. Watts): Use the identification between states $|a\rangle \in \mathcal{V}_j \otimes \mathcal{V}_{\bar{j}}$ and operators $X_a \in \mathrm{Hom}(\mathcal{V}_{\bar{j}}, \mathcal{V}_j)$, namely $|a\rangle = \sum_{n,\bar{n}} a_{n,\bar{n}}|j,n\rangle \otimes |\bar{j},\bar{n}\rangle \leftrightarrow X_a = \sum_{n,\bar{n}} a_{n,\bar{n}}|j,n\rangle\langle\bar{j},\bar{n}|$. Here we make use of the scalar product in $\mathcal{V}_{\bar{j}}$ for which $\bar{L}_{-n} = \bar{L}_n^\dagger$, hence (3.6) means that $L_n X_a = X_a L_n$, i.e. X_a intertwines the action of Vir in the two irreps $\mathcal{V}_j$ and $\mathcal{V}_{\bar{j}}$. By Schur's lemma, this implies that they are equivalent, $\mathcal{V}_j \sim \mathcal{V}_{\bar{j}}$, i.e. that their labels coincide $j = \bar{j}$ and that X_a is proportional to P_j, the projector in $\mathcal{V}_j$. We shall denote $|j\rangle\rangle$ the corresponding state, solution to (3.6).

Since "exponents" $j \in \mathrm{Exp}$ may have some multiplicity, an extra label should be appended to our notation $|j\rangle\rangle$. We omit it for the sake of simplicity. The previous considerations extend with only notational complications to more general chiral algebras and their possible gluing automorphisms Ω. See [14] for more details on these points. Also, in this discussion, I have been a bit cavalier on some points: the fact that these Ishibashi states have no finite norm and thus do not really belong to $\mathcal{H}$, and the use of Schur's lemma in this context would require some justification: See [43] for an alternative and more precise discussion.

The normalization of this "Ishibashi state" requires some care. One first notices that, for $\tilde{q}$ a real number between 0 and 1,

$$\langle\langle j'|\tilde{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|j\rangle\rangle = \delta_{jj'}\chi_j(\tilde{q}) \tag{3.7}$$

up to a constant that we choose equal to 1. It would seem natural to then define the norm of these states by the limit $\tilde{q} \to 1$ of (3.7). This limit diverges, however, and the adequate definition is rather

$$\langle\langle j\|j'\rangle\rangle = \delta_{jj'}S_{1j} \tag{3.8}$$

This comes about in the following way: a natural regularization of the above limit is:

$$\langle\langle j\|j'\rangle\rangle = \lim_{\tilde{q}\to 1} q^{c/24}\langle\langle j'|\tilde{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|j\rangle\rangle \tag{3.9}$$

where q is the modular transform of $\tilde{q} = e^{-2\pi i/\tau}$, $q = e^{2\pi i\tau}$. In a ("unitary") theory in which the identity representation (denoted 1) is the one with the smallest conformal weight, show that in the limit $q \to 0$, the r.h.s. of (3.9) reduces to (3.8). In non unitary theories, this limiting procedure fails, but we keep (3.8) as a definition of the new norm.

At the term of this study, we have found a basis of solutions to the constraint (3.6) on boundary states, and it is thus legitimate to expand the two states attached to the two boundaries of our domain as

$$|a\rangle = \sum_{j\in\mathcal{E}} \frac{\psi_a^j}{\sqrt{S_{1j}}}\,|j\rangle\rangle \tag{3.10}$$

with coefficients denoted ψ_a^j, and likewise for $|b\rangle$. We define an involution $a \to a^*$ on the boundary states by $\psi_{a^*}^j = \psi_a^{j^*} = (\psi_a^j)^*$, (recall that $j \to j^*$ is an involution in $\mathcal{E}$). One may show [44] that it is natural to write for the conjugate state

$$\langle b| = \sum_{j\in\mathcal{E}} \langle\langle j| \frac{\psi_{b^*}^j}{\sqrt{S_{1j}}} \ . \tag{3.11}$$

As a consequence

$$\langle b\|a\rangle = \sum_{j\in\mathcal{E}} \frac{\psi_a^j \left(\psi_b^j\right)^*}{S_{1j}} \langle\langle j\|j\rangle\rangle = \sum_{j\in\mathcal{E}} \psi_a^j \left(\psi_b^j\right)^* \tag{3.12}$$

so that the orthonormality of the boundary states is equivalent to that of the ψ's.

3.3. Cardy equation

Let us return to the annulus $1 \le |\zeta| \le e^{2\pi L/T}$ considered in last subsection, or equivalently to the cylinder of length L and perimeter T, with boundary conditions (b.c.) a and b on its two ends. Following Cardy [45], we shall compute its partition function $Z_{b|a}$ in two different ways. If we regard it as resulting from the evolution between the boundary states $|a\rangle$ and $\langle b|$, with $\tilde{q}^{\frac{1}{2}} = e^{-2\pi L/T}$, we find

$$\begin{aligned} Z_{b|a} &= \langle b|(\tilde{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|a\rangle = \sum_{j,j'\in\mathcal{E}} \frac{\left(\psi_b^j\right)^* \psi_a^{j'}}{S_{1j}} \langle\langle j|\tilde{q}^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})}|j'\rangle\rangle \\ &= \sum_{j\in\mathcal{E}} \psi_a^j \left(\psi_b^j\right)^* \frac{\chi_j(\tilde{q})}{S_{1j}} \ . \end{aligned} \tag{3.13}$$

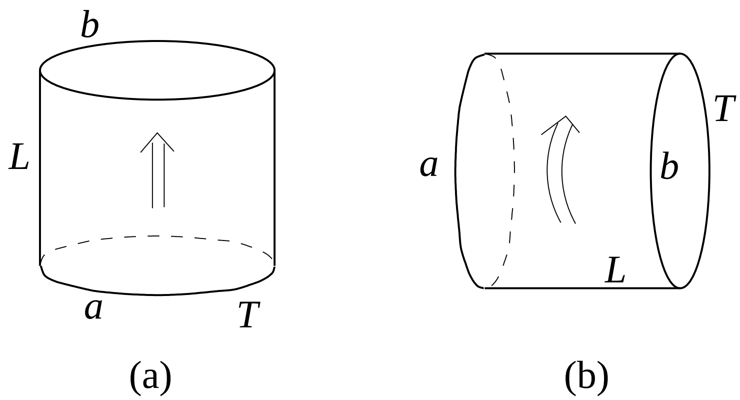


Fig. 4: Two alternative computations of the partition function $Z_{b|a}$: (a) on the cylinder, between the boundary states $|a\rangle$ and $\langle b|$, (b) as a periodic time evolution on the strip, with boundary conditions a and b.

On the other hand, if we regard it as resulting from the periodic "time" evolution on the strip with b.c. a and b, using the decomposition (3.3) of the Hilbert space $\mathcal{H}_{ba}$, and with $q = e^{-\pi T/L}$

$$Z_{b|a}(q) = \sum_{i\in\mathcal{I}} n_{ib}{}^{a} \chi_i(q) \ . \tag{3.14}$$

(Note that string theorists would refer to these two situations as (a): the tree approximation of the propagation of a closed string; (b) the one-loop evolution of an open string). Performing a modular transformation on the characters $\chi_j(\tilde q) = \sum_i S_{ji^*}\chi_i(q)$ in (3.13), and identifying the coefficients of χ_i yields

$$n_{ia}{}^{b} = \sum_{j\in\mathcal{E}} \frac{S_{ij}}{S_{1j}}\, \psi_a^j \left(\psi_b^j\right)^* \ , \tag{3.15}$$

a fundamental equation for our discussion that we refer to as Cardy equation [45]. In deriving this form of Cardy equation, we have made use of the first of the following symmetry properties of $n_{ia}{}^{b}$

$$n_{ia}{}^{b} = n_{i^*b}{}^{a} \tag{3.16}$$

which follow from the previous relations on ψ and of the symmetries of S.

(Comment: this identification of coefficients of specialized characters is in general not justified, as the $\chi_i(q)$ are not linearly independent. As in sect. 2, it is better to generalize the previous discussion, in a way which introduces non-specialized –and linearly independent– characters. This has been done in [14] for the case of CFTs with a current algebra. Unfortunately, little is known about other chiral algebras and their non-specialized characters.)

Let us stress that in (3.15), the summation runs over $j \in \mathcal{E}$, i.e. this equation incorporates some information on the spectrum of the theory "in the bulk", i.e. on the modular invariant partition function (2.6).

Cardy equation (3.15) is a non linear constraint relating a priori unknown complex coefficients ψ^j to integer multiplicities $n_{ia}{}^{b}$. We need additional assumptions to exploit it.

We shall thus assume that

- we have found an orthonormal set of boundary states $|a\rangle$, i.e. satisfying

$$(n_1)_a{}^{b} = \sum_{j\in\mathcal{E}} \psi_a{}^{j} (\psi_b{}^{j})^* = \delta_{ab} \ ; \tag{3.17}$$

- we have been able to construct a *complete* set of such boundary states $|a\rangle$

$$\sum_a \psi_a{}^{j} (\psi_a{}^{j'})^* = \delta_{jj'} \ . \tag{3.18}$$

Note that the second assumption –which looks far from obvious to me– implies that

$$\#\ \text{boundary states} = \#\ \text{independent Ishibashi states} = |\mathcal{E}| \ .$$

3.4. Representations of the fusion algebra and graphs

Return to Cardy equation (3.15) and observe that it gives a decomposition of the matrices n_i, defined by $(n_i)_a{}^b = n_{ia}{}^b$, into their orthonormal eigenvectors ψ and their eigenvalues S_{ij}/S_{1j}. Observe also that as a consequence of Verlinde formula (1.28), these eigenvalues form a one-dimensional representation of the fusion algebra

$$\frac{S_{i\ell}}{S_{1\ell}}\,\frac{S_{j\ell}}{S_{1\ell}} = \sum_{k\in\mathcal{I}} \mathcal{N}_{ij}{}^k\, \frac{S_{k\ell}}{S_{1\ell}}\,, \qquad \forall i,j,\ell \in \mathcal{I}\ . \tag{3.19}$$

Hence the matrices n_i also form a representation of the fusion algebra

$$n_i\, n_j = \sum_{k\in\mathcal{I}} \mathcal{N}_{ij}{}^k\, n_k \tag{3.20}$$

and they thus commute. Moreover, as we have seen above, they satisfy $n_1 = I$, $n_i^T = n_{i^*}$.

Conversely, consider any $\mathbb{N}$-valued matrix representation of the Verlinde fusion algebra n_i, such that $n_i^T = n_{i^*}$. Since the algebra is commutative, $[n_i, n_i^T] = [n_i, n_{i^*}] = 0$. The $\{n_i\}$ form a set of *normal* matrices, hence are diagonalizable in a common orthonormal basis. Their eigenvalues are known to be of the form S_{ij}/S_{1j}. They may thus be written as in (3.15). Thus any such $\mathbb{N}$-valued matrix representation of the Verlinde fusion algebra gives a (complete orthonormal) solution to Cardy's equation.

Conclusion:

$\mathbb{N}$-valued matrix representation of the fusion algebra

$\Longleftrightarrow$ Complete, orthonormal solution of Cardy equation

Moreover, since $\mathbb{N}$-valued matrices are naturally interpreted as graph adjacency matrices, graphs appear naturally!

The relevance of the fusion algebra in the solution of Cardy equation had been pointed out by Cardy himself for diagonal theories [45] and foreseen in general in [29] with no good justification; the importance of the assumption of completeness of boundary conditions was first stressed by Pradisi et al [46].

3.5. The case of $\widehat{sl}(2)$ WZW theories

Problem: *Classify all $\mathbb{N}$-valued matrix reps of $\widehat{s\ell}(2)_k$ fusion algebra with k fixed.*
The algebra is generated recursively by n_2

$$n_1 = I, \qquad n_2\, n_i = n_{i+1} + n_{i-1}, \quad i = 2, \ldots, k \tag{3.21}$$

$$S \text{ real } \Rightarrow\ n_i = n_i^T\ .$$

Even though ψ^j and $\mathcal{E}$ are yet unknown we know from (1.25) that n_2 has eigenvalues of the form

$$\gamma_j = \frac{S_{2j}}{S_{1j}} = 2\cos\frac{\pi j}{k+2}, \quad j \in \mathcal{E}\ . \tag{3.22}$$

But as discussed already in sect. 2.2, $\mathbb{N}$-valued matrices G with spectrum $|\gamma| < 2$ have been classified. They are the adjacency matrices either of the A-D-E Dynkin diagrams or of the "tadpoles" $T_n = A_{2n}/\mathbb{Z}_2$. Thus as a consequence of equation (3.15) alone, for a $\widehat{sl}(2)$ theory at level k, the possible boundary conditions are in one-to-one correspondence with the vertices of one of these diagrams G, with Coxeter number $h = k+2$. If we remember, however, that the set $\mathcal{E}$ must appear in one of the modular invariant torus partition functions, the case $G = T_n$ has to discarded, and we are left with ADE. (Up to this last step, this looks like the simplest route leading to the ADE classification of $\widehat{sl}(2)$ theories.) We thus conclude that for each $\widehat{sl}(2)$ theory classified by a Dynkin diagram G of ADE type

$$
\begin{aligned}
\mathcal{E} &= \mathrm{Exp}(G), \qquad \dim(n_i) = |\mathcal{E}| = |G| \\
\text{complete orthonormal b. c. } &= a, b, \cdots \ : \ \text{vertices of } G \\
n_2 &= \text{ adjacency matrix of } G \\
n_i &= \text{“}i\text{-th fused adjacency matrix” of } G \\
\psi^j &= \text{eigenvector of } n_2 \text{ with eigenvalue } \gamma_j
\end{aligned}
$$

One checks indeed that the matrices n_i, given by equation (3.15), together with (1.25), have only non negative integer elements. Let us pause a little to examine the remarkable properties of these matrices which seem to play an ubiquitous role...

3.6. Side remark: the $\widehat{sl}(2)$ "intertwiners"

Let G be a given Dynkin diagram of ADE type, with Coxeter number h. We want to look more closely at properties of the matrices n_i just defined. Explicitly, using (1.25),

$$
{n_{ia}}^b = \sum_{\ell \in \mathrm{Exp}(G)} \frac{\sin \frac{\pi \ell i}{h}}{\sin \frac{\pi \ell}{h}} \psi_a^\ell \psi_b^{\ell *} \ . \tag{3.24}
$$

As stated above, all their matrix elements are non negative integers, and they form a representation of the fusion algebra. But also, regarded as rectangular matrices for fixed a, they satisfy an intertwining property $An = nG$ with the adjacency matrix A of the Dynkin diagram A_{h-1}. More explicitly

$$
\sum_{j \in A_{h-1}} A_{ij} {n_{ja}}^b = \sum_{c \in G} {n_{ia}}^c G_{cb} \tag{3.25}
$$

These matrices n have made repeated appearances in various contexts.

(1) In CFT: For an appropriate choice of a subset T of the vertices of G [(4)] and in particular of a special vertex denoted 1, $1 \in T$, one may write the torus partition functions of type $A, D_{\mathrm{even}}, E_6, E_8$ in the form

$$
Z_{\mathrm{torus}} = \sum_{a \in T} |\hat{\chi}_a|^2 \tag{3.26}
$$

[(4)] the set of ambichiral vertices in the language of A. Ocneanu

where $\hat\chi_a := \sum_i n_{i1}{}^a \chi_i$ are combinations of characters of the original algebra, interpreted as characters of a larger "extended" algebra (see sect. 2.2 *in fine*). This formula, originally found empirically in [29] and justified later in a variety of cases in [38], has been extended to the missing cases $D_{2\ell+1}, E_7$ by use of a relative twist between the right and left characters $\hat\chi$ pertaining to the $A_{4\ell-1}$ and D_{10} cases, respectively [39], in agreement with the general result of [37,9] recalled in sect 2.2 *in fine*. The general formula is thus

$$Z_{\text{torus}} = \sum_{a\in T} \hat\chi_a(q) \left(\hat\chi_{\zeta(a)}(q)\right)^* \ . \tag{3.27}$$

Here ζ is an automorphism of the fusion algebra of the representations of the extended algebra labelled by $a \in T$. This formula has also received a new interpretation in the light of the work of [31][40]: see D. Evans' lectures at this school, and compare his expression $\langle \alpha_i^+, \alpha_j^- \rangle$ for the matrix N_{ij}, with that coming from (3.27) $N_{ij} = \sum_{a\in T} n_{a1}{}^i \, n_{\zeta(a)1}{}^j$.

(2) Lattice models, graphs, operator algebras. These same matrices n also give the decomposition of representations of the Temperley-Lieb algebra on the space of paths from a to b on the graph G onto the irreducible ones given by the paths from 1 to i on graph A_{h-1} [47]

$$R_a^{(G)b} = \oplus_i n_{ia}{}^b R_1^{(A)i}$$

Another manifestation of this is that they give the counting of "essential" paths [39]. See Appendix B for details.

(3) Kostant polynomials in McKay's correspondence: More surprisingly, maybe, these matrices also appear in the explicit expressions of the so-called Kostant polynomials, in the context of McKay correspondence, [48]: see Appendix A. In that context too, the matrices n have received a very neat group theoretical interpretation by Dorey, in terms of the action of the Coxeter element on simple roots [49].

(4) Finally, they have lately made repeated appearances in the context of integrable theories, e.g. in the S-matrices of affine Toda theories [50], or in the excitation spectrum of integrable lattice models [51].

3.7. Other cases

It should be clear that the situation that we have described in detail for $sl(2)$ extends to all RCFTs. The matrices n_i solutions to Cardy equation are the adjacency matrices of graphs. In the case of $\widehat{sl}(N)$, it is sufficient to supply the $(N-1)$ fundamental matrices $n_{\left.\begin{smallmatrix}\square\\ \vdots\\ \square\end{smallmatrix}\right\}p}$, $p = 1, \cdots, N-1$, to determine all of them. The fact that all n_i then have non negative integer elements is non trivial. By Cardy equation again, they satisfy a very restrictive spectral property: their eigenvalues must be of the form S_{ij}/S_{1j}, when j runs over the set $\mathcal{E}$, i.e. the diagonal part of the modular invariant. We have thus found a justification of the empirical association between RCFTs and graphs, (see sect 2.4), and we have found a physical interpretation of the matrix element $n_{ia}{}^b$ as the multiplicity of representation i in the presence of the boundary conditions a and b.

The program of classifying these graphs/boundary conditions has been completed only in a few cases: $\widehat{sl}(2)$ as discussed above, $\widehat{sl}(3)$ through a combination of Gannon's work and the recent work of Ocneanu [32], see sect. 2.3; $\widehat{sl}(N)_1$ [14], where the results match those obtained in the study of modular invariants [12]: the graphs turn out to be star polygons. The case of minimal ($c < 1$) models has also been fully analysed [14].

3.8. Other algebraic features

The identification of the allowed boundary conditions with the determination of the multiplicities and of the associated graphs is just the beginning of the story. A more elaborate discussion of BCFT should include a study of the operator algebra in the presence of a boundary. This is an important, interesting and lively subject, a general understanding of which is still missing. It is also beyond the scope of these introductory lectures. Let me only mention that in that study, it appears that a triplet of algebras (n_i, M_i, N_a) plays a key role. Let

$$\begin{aligned} {n_{ia}}^b &= \sum_{\ell\in\mathcal{E}} \frac{S_{i\ell}}{S_{1\ell}}\, \psi_a^\ell\,(\psi_b^\ell)^* \\ {M_{ij}}^k &= \sum_{a\in G} \frac{\psi_a^i \psi_a^j (\psi_a^k)^*}{\psi_a^1} \\ {\hat{N}_{ab}}{}^c &= \sum_{\ell\in\mathcal{E}} \frac{\psi_a^\ell \psi_b^\ell (\psi_c^\ell)^*}{\psi_1^\ell}\ . \end{aligned} \tag{3.28}$$

The first has already been encountered. In the definition of the second ("Pasquier algebra"), the positivity of the components of the Perron-Frobenius eigenvector ψ^1 is crucial. For the third, one assumes as above the existence of a special vertex denoted 1, such that all ${\psi_1}^\ell \neq 0$. Note that as matrices, the n, M and $\hat{N}$ satisfy

$$\begin{aligned} n_i n_j &= \sum_k {\mathcal{N}_{ij}}^k n_k \\ M_i M_j &= \sum_k {M_{ij}}^k M_k \\ \hat{N}_a \hat{N}_b &= \sum_c {\hat{N}_{ab}}{}^c \hat{N}_c\ . \end{aligned} \tag{3.29}$$

The role of the first has just been discussed. In most known cases the ${\hat{N}_{ab}}{}^c$ turn out to be integers. In the type I cases, where they are non negative, they seem to describe a certain fusion algebra, generalised to a class of a yet ill-defined class of "twisted" representations of the extended algebra. Restricted to the subset $a \in T$ (see sect. 3.6 (1)), it reduces to the ordinary fusion of ordinary representations of the extended algebra. One may show that the graph adjacency matrices n_i are linear combinations of the $\hat{N}_a$, and the algebra of the latter may be called a graph fusion algebra. This graph fusion algebra is the dual of the Pasquier algebra in the sense of the theory of *C-algebras* [52],[36],[38]. In the context

of BCFT, the Pasquier algebra is deeply connected with the properties of the so-called bulk-boundary coefficients, which describe the coupling of bulk and boundary operators. For more details, I refer the interested reader to [14] and to the many references quoted there.

I have been particularly sloppy on references in this last section. I should mention that over the last ten years, this subject of boundary CFT has received important contributions by many, in particular Saleur and Bauer, Cardy and Lewellen, Affleck and Ludwig, Affleck and Oshikawa, and Saleur, Pradisi, Sagnotti and Stanev, Recknagel and Schomerus, Fuchs and Schweigert, and Runkel: these references may be found in [14]. Additional recent references are by Huiszoon, Schellekens and Sousa, by Gannon, by Felder, Fröhlich, Fuchs and Schweigert, and many others.

4. Lattice integrable realizations

It should also be mentionned that parallel to the conformal field theoretic discussion sketched in these notes, there exists a discussion of lattice integrable models, the so-called face, or height, or RSOS, models. There the Yang-Baxter is realised through a representation of the Temperley-Lieb algebra, or of some other quotient of the Hecke algebra, on the space of paths on a graph: for example the Pasquier models [27] in the simplest case of $sl(2)$, or their higher rank generalizations. Finally, boundaries may be introduced without spoiling integrability, through a careful determination of the boundary Boltzmann weights, satisfying the Boundary Yang-Baxter Equation [53].

Through these different approaches we can see the various facets of a beautiful common algebraic structure…

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Appendix A. The McKay correspondence

The following presents details on the McKay correspondence, the Kostant polynomials and their connection with "intertwiners" of ADE type. The proof of the latter is a slightly expanded version of what appeared in a review article by Di Francesco [36].

Let Γ be a subgroup of $SU(2)$. We denote by (a) its irreducible representations, among which (0) is the identity representation; (f), the "fundamental", is the two-dimensional representation of Γ inherited from that of $SU(2)$. (It may be irreducible or reducible, depending on Γ). The fundamental observation of McKay [24] is that if we tensor product (a) by (f) and decompose it on irreps

$$(f) \otimes (a) = \oplus_b \widehat{G}_{ab}\,(b) \tag{A.1}$$

we find that $\widehat{G}_{ab}$ is the adjacency matrix of an affine Dynkin diagram $\widehat{G}$, thus canonically associated with Γ, according to Table 3 of sect 2.2. In the following, we shall restrict to the subgroups Γ such that $\hat{G}$ is bi-colourable, namely $\mathcal{C}_{2n}, \mathcal{D}_n, \mathcal{T}, \mathcal{O}, \mathcal{I}$.

We want to see how the irreps of $SU(2)$ decompose onto the irreps of Γ: let $[n]$ be the $(n+1)$-dimensional representation of $SU(2)$ restricted to Γ (in particular $[0]$ is the identity representation, and $[1] = (f)$, see above. Let

$$[n] = \oplus_a N_{na}\,(a) \tag{A.2}$$

with a generating function of the multiplicities N written as

$$\begin{aligned} F(t) &= \sum_{n=0}^{\infty} t^n\,[n] = \sum_{a,n} t^n N_{na}\,(a) \\ &= \sum_{\substack{a \\ \text{irreps of }\Gamma}} F_a(t)\,(a)\ . \end{aligned} \tag{A.3}$$

One writes easily recursion formulae

$$\begin{aligned} [1] \otimes F(t) &= \sum_a F_a(t)\,(a) \otimes (f) \\ &= \sum_{n=0}^{\infty} t^n\ ([n+1] + [n-1]) = \left(t + \frac{1}{t}\right) F(t) - \frac{(0)}{t} \end{aligned} \tag{A.4}$$

.

We evaluate this by taking its character on conjugation classes C_i of Γ:

$$\sum_a F_a(t)\chi_a(C_i)\chi_f(C_i) = \left(t + \frac{1}{t}\right) \sum_a F_a(t)\chi_a(C_i) - \frac{1}{t} \tag{A.5}$$

hence

$$\sum_a F_a(t)\chi_a(C_i) = \frac{1}{1 + t^2 - t\chi_f(C_i)}\ , \tag{A.6}$$

or, using the orthogonality of characters $\sum_i\ |C_i|\,\chi_a(C_i)\chi_b^*(C_i) = |\Gamma|\,\delta_{ab}$:

$$F_a(t) = \sum_i \frac{|C_i|}{|\Gamma|}\,\frac{\chi_a^*(C_i)}{1 - t\chi_f(C_i) + t^2}\ . \tag{A.7}$$

The explicit result has been worked out by Kostant [48]. He found

$$F_a(t) = \frac{p_a(t)}{(1 - t^{\mathbf{a}})(1 - t^{\mathbf{b}})} \tag{A.8}$$

where $p_a(t)$ is a polynomial in t of degree less or equal to h (h is the Coxeter number of the *finite* Dynkin diagram G associated with $\widehat{G}$, $\mathbf{a}$ and $\mathbf{b}$ are two integers satisfying $\mathbf{ab} = 2|\Gamma|$, $\mathbf{a} + \mathbf{b} = h + 2$, for example 6 and 8 for E_6).

Exercise : prove that in (A.7) only $|\Gamma|$-th roots of unity appear as poles in t, hence that $\mathbf{a}$ and $\mathbf{b}$ must divide $|\Gamma|$.

Γ	$\mathcal{C}_{2n}$	$\mathcal{D}_n$	$\mathcal{T}$	$\mathcal{O}$	$\mathcal{I}$
$\|\Gamma\|$	$2n$	$4n$	24	48	120
$(\mathbf{a}, \mathbf{b})$	$(2, 2n)$	$(4, 2n)$	$(6, 8)$	$(8, 12)$	$(12, 20)$
h	$2n$	$2n + 2$	12	18	30
G	A_{2n-1}	D_{n+2}	E_6	E_7	E_8

Let us plug the form above in the recursion formula, getting rid of the denominator $(1 - t^{\mathbf{a}})(1 - t^{\mathbf{b}})$

$$\begin{aligned}\left(t + \frac{1}{t}\right) \sum_a p_a(t)(a) - \frac{(0)}{t}(1 - t^{\mathbf{a}})(1 - t^{\mathbf{b}}) &= \sum_a p_a(t)\,(a) \otimes [1] \\ &= \sum_{a,b} p_b(t)\widehat{G}_{ab}\,(a)\ . \end{aligned} \tag{A.9}$$

Denote by G_{ab} the adjacency matrix of the *ordinary* Dynkin diagram, obtained from $\widehat{G}$ by removing the node called 0. Then identifying in (A.9) the coefficient of $(a) \neq (0)$ gives

$$a \neq 0 \qquad \left(t + \frac{1}{t}\right) p_a(t) = \sum_{b \neq 0} p_b(t) G_{ba} + p_0(t)\widehat{G}_{a0}\ . \tag{A.10}$$

Following Kostant, write $p_a(t) = \sum_{n=0}^{h} p_{a\,n}\, t^n$, $p_{a\,n} = p_{a\,h-n}$, $p_0(t) = 1 + t^h$. Eq. (A.10) implies that $p_{a\,0} = p_{a\,h} = 0$ for $a \neq 0$, and thus

$$\begin{aligned}\left(t + \frac{1}{t}\right) p_a(t) &= \sum_{n=1}^{h-1} p_{a\,n}\,(t^{n+1} + t^{n-1}) \\ &= \sum_{n=0}^{h} (p_{a\,n+1} + p_{a\,n-1})\,t^n \qquad \text{with } p_{a\,-1} = p_{a\,h+1} \equiv 0 \\ &= \sum_{m,n=1}^{h-1} t^n A_{nm} p_{a\,m} + t^0 p_{a\,1} + t^h p_{a\,h-1} \end{aligned} \tag{A.11}$$

where A_{nm} is the adjacency matrix of type A. Compare the r.h.s. of equations (A.10) and (A.11). Since degree$(p_a) < h$ for $a \neq 0$, the two extra terms in (A.11) have to be identified with $p_0(t)\widehat{G}_{a0} = (1 + t^h)\widehat{G}_{a0}$ in (A.10), hence

$$p_{a\,1} = p_{a\,h-1} = \widehat{G}_{a0}\ , \tag{A.12}$$

while the identification of the coefficient of the terms t^n, $1 \le n \le h-1$ yields

$$\sum_{m=1}^{h-1} A_{nm} p_{a\,m} = \sum_{b\neq 0} p_{b\,n} G_{ba} \ . \tag{A.13}$$

This may be read as a recursion formula determining uniquely $p_{a\,n+1} = p_{a\,n-1} + \sum_b G_{ba} p_{b\,n}$, starting from $p_{a\,1} = \widehat{G}_{a0}$. But this also proves that $p_{a\,m}$ is an intertwiner between the A and the G Dynkin diagrams. Its expression in terms of the matrices of sect. 3.6

$$p_{a\,m} = \sum_b \widehat{G}_{0b} {n_{mb}}^a \tag{A.14}$$

follows from the observation that both satisfy the boundary conditions (A.12).

This completes our proof of the statement (3) in sect 3.6, namely that Kostant polynomials are given by the intertwiners n:

$$p_a(t) = \sum_{m=1}^{h-1} \sum_b \widehat{G}_{0b} {n_{m\,b}}^a \, t^m \ , \qquad a \neq 0 \ . \tag{A.15}$$

Appendix B. The counting of essential paths

The following gives a slight variant of a proof given by Ocneanu [39], that the entry ${n_{na}}^b$ of the intertwiners of ADE type gives the number of "essential paths" on the diagram under consideration.

Given a graph of ADE type, consider the set of paths of length $n \ge 0$ starting from the vertex a and ending at vertex b, $(a_0 = a, a_1, \cdots, a_n = b)$. Consider the linear span $\mathcal{P}_{ab}^{(n)}$ of these paths. Define the operator Δ_i, the *contraction operator at step* i, by

$$\Delta_i\,(a_0 = 1, a_1, \cdots, a_n) = (a_0 = 1, a_1, \cdots, a_{i-1}, a_{i+2}, \cdots a_n)\, \delta_{a_{i-1}, a_{i+1}} \tag{B.1}$$

i.e. Δ_i gives zero if the path doesn't backtrack. Each Δ_i maps $\mathcal{P}_{ab}^{(n)}$ into $\mathcal{P}_{ab}^{(n-2)}$. Then define the subspace $\mathcal{E}_{ab}^{(n)}$ of essential paths from a to b of length n as those that are in the kernel of all Δ_i in $\mathcal{P}_{ab}^{(n)}$, for $i = 1, \cdots n-1$. It is important that this concept is defined in the vector space, because for two paths that both backtrack and would yield the same contracted path, their *difference* is in the kernel. This is typically what happens at the "fork" of a graph D: the two paths that "bounce" off the two end points have a difference that is essential. (As a side remark, it is more suitable to normalise differently the contraction operator [39]. This does not affect the dimension of its kernel but the explicit form of its null vectors is changed.)

As proved by Ocneanu [39], the counting of essential paths of length n with fixed ends a, b (i.e. the dimension of $\mathcal{E}_{ab}^{(n)}$) is given by the intertwiner ${n_{n+1\,a}}^b$. In particular the length of essential paths is bounded by the Coxeter number -1. I give here a proof slightly

different from that of Ocneanu, in which I show that the recursion formula for essential paths is just the same as that for the intertwiners, namely (3.21).

Denoting by $a:b$ the property of a and b to be neighbours on the graph, one has

$$\begin{aligned}
\mathcal{E}_{ab}^{(n)} &= \cap_{i=1}^{n-1} (\ker \Delta_i)_{\mathcal{P}_{ab}^{(n)}} \\
&= \Big(\oplus_{c:b} \cap_{i=1}^{n-2} (\ker \Delta_i)_{\mathcal{P}_{ac}^{(n-1)}} \Big) \cap (\ker \Delta_{n-1})_{\mathcal{P}_{ab}^{(n)}} \\
&= (\ker \Delta_{n-1})_{\oplus_{c:b} \cap_{i=1}^{n-2} (\ker \Delta_i)_{\mathcal{P}_{ac}^{(n-1)}}} \ .
\end{aligned} \tag{B.2}$$

Then consider the *image of* Δ_{n-1} in the space $\oplus_{c:b} \cap_{i=1}^{n-2} (\ker \Delta_i)_{\mathcal{P}_{ac}^{(n-1)}}$ i.e. in the subspace of $\mathcal{P}_{ab}^{(n)}$ of paths essential up to site $n-2$ (the paths of those kernels are continued from length $n-1$ to length n by" adding" the last step $c-b$). This subspace is $\mathcal{E}_{ab}^{(n-2)}$, since for a linear combination of paths p

$$\Delta_{n-1} \sum_p c_p [p = (a_0, \cdots, a_{n-1}(p), a_n = b)] = \sum_p c_p (a_0, \cdots, a_{n-2}(p)) \delta_{a_{n-2},b} \tag{B.3}$$

which is a generic element of $\mathcal{E}_{ab}^{(n-2)}$. Hence the dimension of its kernel in that space i.e. the dimension of the r.h.s. of (B.2) is

$$\mathcal{N}_{ab}^{(n)} = \dim \mathcal{E}_{ab}^{(n)} = \sum_{c:b} \dim \mathcal{E}_{ac}^{(n-1)} - \mathcal{N}_{ab}^{(n-2)} \ . \tag{B.4}$$

Thus the numbers $\mathcal{N}$ satisfy the same recursion formula (3.21) as the n's, and the same boundary conditions, hence are identical. QED

References

[1] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Nucl. Phys.* **B 241** (1984) 333-380.

[2] V.G. Kac, *Lect. Notes in Phys.* **94** (1979) 441-445;
B.L. Feigin and D.B. Fuchs, *Funct. Anal. and Appl.* **16** (1982) 114-126; *ibid.* **17** (1983) 241-242.

[3] V. Kac, *Infinite dimensional algebras*, Cambridge University Press; V.G. Kac and D.H. Peterson, *Adv. Math.* **53** (1984) 125-264;
J. Fuchs, *Affine Lie Algebras and Quantum Groups*, Cambridge University Press.

[4] P. Goddard, A. Kent and D. Olive, *Comm. Math. Phys.* **103** (1986) 105-119.

[5] P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory*, Springer Verlag 1997.

[6] E. Verlinde, *Nucl. Phys.* **B300** [FS22] (1988) 360-376.

[7] J. Cardy, *Nucl Phys* **B 270** (1986) 186-204.

[8] H. Sonoda, *Nucl. Phys.* **B281** (1987) 546-572; **B 284** (1987) 157-192.

[9] G. Moore and N. Seiberg, *Nucl. Phys.* **B313** (1989) 16-40; *Comm. Math. Phys.* **123** (1989) 177-254.

[10] A. Cappelli, C. Itzykson and J.-B. Zuber, *Nucl. Phys.* **B280** [FS18] (1987) 445-465; *Comm. Math. Phys.* **113** (1987) 1-26 ;
A. Kato, *Mod. Phys. Lett.* **A2** (1987) 585-600.

[11] T. Gannon, *Comm. Math. Phys.* **161** (1994) 233-263; *The Classification of* $SU(3)$ *Modular Invariants Revisited*, hep-th 9404185.

[12] C. Itzykson, Nucl. Phys. (Proc. Suppl.) **5B** (1988) 150-165 ;
P. Degiovanni, Comm. Math. Phys. **127** (1990) 71-99.

[13] T. Gannon, *The monstruous moonshine and the classification of CFT*, `hep-th 9906167`.

[14] R.E. Behrend, P.A. Pearce, V.B. Petkova and J.-B. Zuber, *Boundary conditions in RCFT*, *Nucl. Phys.* to **re**appear, hep-th 9908036.

[15] M. Hazewinkel, W. Hesselink, D. Siersma and F.D. Veldkamp, *Nieuw Archief voor Wiskunde*, **3** XXV (1977) 257-307.

[16] N. Bourbaki, *Groupes et Algèbres de Lie*, chap. 4–6, Masson 1981; J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer Verlag, New York, 1972.

[17] H.S.M. Coxeter, *Ann. Math.* **35** (1934) 588-621;
J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Univ. Pr. 1990.

[18] For a review and references, see P. Slodowy, *Platonic solids, Kleinian singularities and Lie groups*, Lect. Notes in Math. **1008** (1983) 102-138, Springer; *Algebraic groups and resolutions of Kleinian singularities*, RIMS-1086, Kyoto 1996.

[19] V.I. Arnold, S.M. Gusein-Zaide and A.N. Varchenko, *Singularities of differential maps*, Birkhäuser, Basel 1985.

[20] P. Gabriel, *Manusc. Math.* **6** (1972) 71-103.

[21] E. Hille, *Ordinary Differential Equations in the Complex Domain*, J. Wiley, 1976.

[22] V.F.R. Jones, *Invent. Math.* **72** (1983) 1-25.

[23] F.M. Goodman, P. de la Harpe and V.F.R. Jones, *Coxeter Graphs and Towers of Algebras*, Springer-Verlag, Berlin (1989).

[24] J. McKay, *Proc. Symp. Pure Math.* **37** (1980) 183-186.

[25] J.-L. Brylinski, *A correspondance dual to McKay's*, `alg-geom 9612003`.

[26] T. Eguchi and S.K. Yang, *Mod. Phys. Lett.* **A5** (1990) 1693-1701 ;
R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. **B352** 59-86 (1991): in *String Theory and Quantum Gravity*, proceedings of the 11990 Trieste Spring School, M. Green et al. *eds.*, World Sc. 1991.

[27] V. Pasquier, *Nucl. Phys.* **B285** [FS19] (1987) 162-172 ;
V. Pasquier, *J. Phys.* **A20** (1987) 5707-5717.

[28] W. Lerche, C. Vafa and N. Warner, *Nucl. Phys.* **B 324** (1989) 427-474 ;
E. Martinec, in *Criticality, catastrophes and compactifications*, in *Physics and mathematics of strings*, V.G. Knizhnik memorial volume, L. Brink, D. Friedan and A.M. Polyakov eds., World Scientific 1990 ;
P. Howe and P. West, *Phys. Lett.* **B223** (1989) 377-385; *ibid.* **B227** (1989) 397-405.

[29] P. Di Francesco and J.-B. Zuber, *Nucl. Phys.* **B338** (1990) 602-646.

[30] N. Sochen, *Nucl. Phys.* **B360** (1991) 613-640.

[31] F. Xu, *Comm. Math. Phys.* **192** (1998) 349-403.

[32] A. Ocneanu, lectures at this school.

[33] J.-B. Zuber, *C-algebras and their applications to reflection groups and conformal field theories*, lectures at RIMS, Kyoto December 1996, `hep-th 9707034`.

[34] A. Ocneanu, "Quantized groups, string algebras and Galois theory for algebras", in *Operator Algebras and Applications*, D. Evans and M. Takesaki eds, 1988, pp. 119-172 ;
D. Evans and Y. Kawahigashi, *Publ. RIMS, Kyoto Univers.* **30** (1994) 151-166.

[35] V. Pasquier, *Modèles Exacts Invariants Conformes*, Thèse d'Etat, Orsay, 1988.

[36] P. Di Francesco and J.-B. Zuber, in *Recent Developments in Conformal Field Theories*, Trieste Conference 1989, S. Randjbar-Daemi, E. Sezgin and J.-B. Zuber eds., World Scientific 1990 ;
P. Di Francesco, *Int. J. Mod. Phys.* **A7** (1992) 407-500.

[37] E. Verlinde and R. Dijkgraaf, *Nucl. Phys. (Proc. Suppl.)* **5B** (1988) 87-97.

[38] V.B. Petkova and J.-B. Zuber, *Nucl. Phys.* **B 463** (1996) 161-193, `hep-th 9510175`; `hep-th 9701103`.

[39] A. Ocneanu, *Paths on Coxeter Diagrams*, in *Lectures on Operator Theory* Fields Institute Monographies, Rajarama Bhat et al edrs, AMS 1999.

[40] J. Böckenhauer and D.E. Evans, *Comm.Math. Phys.* **197** (1998) 361-386; *ibidem* **200** (1999) 57-103, `hep-th 9805023`; *ibidem* **205** (1999) 183-228, `hep-th/9812110`; J. Böckenhauer, D.E. Evans, and Y. Kawahigashi, *Comm.Math. Phys.* **208** (1999) 429-487, `math-oa 9904109`; *ibidem* **210** (2000) 733-784; see also D. Evans' lectures at this school.

[41] J.-B. Zuber, *Generalized Dynkin diagrams and root systems and their folding*, Proccedings of the Taniguchi meeting, Kyoto Dec 1996, *Topological Field Theory, Primitive Forms and Related Topics*, M. Kashiwara, A. Matsuo, K. Saito and I. Satake edrs, Birkhäuser.

[42] S.M. Gusein-Zaide and A.N. Varchenko, *Verlinde algebras and the intersection form on vanishing cycles*, `hep-th 9610058`.

[43] J. Fuchs and C. Schweigert, *Nucl. Phys.* **B 530** (1998) 99-136.

[44] A. Recknagel and V. Schomerus, *Nucl. Phys.* **B 531** (1998) 185-225.

[45] J.L. Cardy, *Nucl. Phys.* **324** (1989) 581-596.

[46] G. Pradisi, A. Sagnotti and Ya.S. Stanev *Phys. Lett.***381** (1996) 97-104.

[47] V. Pasquier and H. Saleur *Nucl. Phys.* **B 330** (1990) 523-556.

[48] B. Kostant, *Proc. Natl. Acad. Sci. USA* **81** (1984) 5275-5277; *Astérisque (Société Mathématique de France)*, (1988) 209-255, *The McKay Correspondence, the Coxeter Element and Representation Theory.*

[49] P. Dorey, *Int. J. Mod. Phys.***8** (1993) 193-208.

[50] H. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, *Nucl. Phys.* **B 338** (1990) 689-746.

[51] B.M. McCoy and W. Orrick *Phys. Lett.* **A 230** (1997) 24-32 ;
M.T. Batchelor and K.A. Seaton, K.A., `cond-mat 9803206`, *Excitations in the diluate A_L lattice model: E_6, E_7 and E_8 mass spectra* ;
J. Suzuki, `cond-mat 9805241`, *Quantum Jacobi-Trudi Formula and E_8 Structure in the Ising Model in a Field.*

[52] E. Bannai, T. Ito, *Algebraic Combinatorics I: Association Schemes*, Benjamin/Cummings (1984).

[53] R.E. Behrend and P.A. Pearce, *J. Phys. A* **29** (1996) 7827-7835; *Int. J. Mod. Phys.* **11** (1997) 2833-2847; *Integrable and Conformal Boundary Conditions for sl(2) A-D-E Lattice Models and Unitary Minimal Conformal Field Theories*, `hep-th hep-th/0006094`.